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Self-adjoint extensions and spectral analysis in the Calogero problem

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Abstract

In this paper, we present a mathematically rigorous quantum-mechanical treatment of a one-dimensional motion of a particle in the Calogero potential αx^{-2} . Although the problem is quite old and well studied, we believe that our consideration based on a uniform approach to constructing a correct quantum-mechanical description for systems with singular potentials and/or boundaries, proposed in our previous works, adds some new points to its solution. To demonstrate that a consideration of the Calogero problem requires mathematical accuracy, we discuss some ‘paradoxes’ inherent in the ‘naive’ quantum-mechanical treatment. Using a self-adjoint extension method, we construct and study all possible self-adjoint operators (self-adjoint Hamiltonians) associated with a formal differential expression for the Calogero Hamiltonian. In particular, we discuss a spontaneous scale-symmetry breaking associated with self-adjoint extensions. A complete spectral analysis of all self-adjoint Hamiltonians is presented.

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1. Introduction

In this paper, we present a mathematically rigorous nonrelativistic quantum-mechanical (QM) treatment of a one-dimensional motion of a particle in the potential field

$$V(x) = \alpha x^{-2}, \quad (1)$$

singular at the origin. The case of $\alpha > 0$ corresponds to repulsion from the origin; the case of $\alpha < 0$ corresponds to attraction to the origin.

Our aim is twofold. First, although the problem is quite old and well studied, both by physicists and mathematicians (see the discussion below), we believe that our consideration adds some new points to its solution. Second, we present another illustration of a uniform approach to constructing a correct QM description for systems with singular potentials

and/or boundaries, proposed in our previous works [1–5]. This description incorporates a proper definition of physical observables as self-adjoint (s.a. in what follows) operators in an appropriate Hilbert space, see, e.g., [6–8], with special attention to possible ambiguities inherent in the description, and the spectral analysis of the observables. The first example of such a description, as applied to a relativistic spin-1/2 particle moving in the Coulomb field of arbitrary charge, was presented in [5].

Starting from the basic papers by Calogero on the exactly solvable one-dimensional QM models [9–11], the potential (1) is conventionally called the Calogero potential, and the problem of a QM description of the system of particles with this pair potential is known as the Calogero problem.

We restrict ourselves to the case of a motion on the semiaxis $\mathbb{R}_+ = [0, \infty)$. The case of the whole axis $\mathbb{R} = (-\infty, \infty)$, or that of a finite interval $[0, a]$, can be considered by the same methods. We only mention that setting up the corresponding quantum mechanics (QM) contains more ambiguity.

The Calogero problem on the semiaxis is of physical significance because it can be considered as the problem of a radial motion of a particle in higher dimensions in the potential field $V(r) \sim 1/r^2$; the variable x is then a radius r , cylindrical or spherical. In particular, this problem is associated with the three-dimensional motion of a charged particle in the magnetic field of an infinitely thin and infinitely long solenoid, in which case $x = r$ is the cylindrical radius, or in the field of a magnetic monopole, in which case $x = r$ is a spherical radius; see, e.g., [12] and references therein. It is also associated with the three-dimensional motion of a polarizable atom in the electric field of an infinitely thin and infinitely long charged wire [13].

The peculiarity of higher-dimension classical mechanics in the case of attraction is that under some initial conditions the particle ‘falls to the center’ in a finite time interval [14], such that the final state at the end of this interval is a position $\mathbf{r} = 0$ and a momentum $\mathbf{p} = \infty$ of an uncertain direction, and the problem arises how to define the motion of the particle after this time interval. In some sense, QM ‘inherits’ these difficulties, although gives them a QM form.

A ‘fall to the center’ manifests itself in the case of $\alpha < -1/4$ as the unboundedness of the energy spectrum from below; see, e.g., [15]. In addition, as was found in the very beginning of QM, the conventional QM methods of finding energy eigenvalues and eigenfunctions fail in this case [16]. By ‘conventional’, we mean the customary methods adopted in physical textbooks and reduced to directly solving the corresponding differential equations with the only requirements of square-integrability for bound-state eigenfunctions (discrete spectrum), local square-integrability at the singularity/boundary, boundedness at infinity, and ‘normalizability to the δ function’ for scattering state eigenfunctions (continuous spectrum). These difficulties are characteristic for all strongly singular attractive potentials like $V(r) \sim \alpha/r^n$, $n \geq 2$, which initially even raised the doubt whether such potentials fall into the realm of QM. A possible way out is to declare that strongly attractive potentials are inadmissible extrapolations of the known physical forces to arbitrarily small distances and therefore have no physical meaning without cutting off the singularity. But then the question arises as to what extent a physical description, in particular, low-energy physics, depends on a cut-off and how a possible ambiguity in the description can be parameterized. Examining strongly singular potentials by themselves, we answer this question to some extent. It is also worth noting that the attractive Coulomb potential is strongly singular for relativistic particles.

The first step in overcoming the above difficulties was due to Case, who noted that a quantum Hamiltonian with a strongly singular attractive potential, in particular, the Calogero

potential with $\alpha < -1/4$, is not defined by the formal differential expression alone, but ‘needs a further specification by requiring a fixed phase for the wavefunctions at the origin’, and the phase is ‘an additional (to the functional form of the potential) parameter’ [17]. This requirement followed from the orthogonality condition for eigenfunctions of bound states with different energy eigenvalues. It is remarkable that the phase is not determined uniquely, so that there exists a one-parameter family of candidates for the quantum Hamiltonian. A formula for the negative spectrum of the Calogero Hamiltonian with $\alpha < -1/4$ and an arbitrary fixed phase was thus first presented. In fact, as we now realize, this was the first formulation of additional asymptotic s.a. boundary conditions at the singularity that specified, nonuniquely, a s.a. Hamiltonian, although self-adjointness (‘hermicity’) was understood as the orthogonality and completeness of eigenfunctions, and the completeness was only declared. The next step was due to Meetz, who pointed out that a proper treatment of singular potentials, in particular, the Calogero problem, requires invoking the theory of s.a. extensions of symmetric operators³, including such notions as deficient subspaces and deficiency indices [19]. Self-adjoint Hamiltonians with singular attractive potentials, and even with some repulsive Calogero potentials, were then specified in terms of the respective deficient subspaces, which introduced an extra parameter; the conjecture by Case was thus confirmed. Proper spectral decompositions of the resulting Hamiltonians were also systematically elaborated⁴. It was also emphasized that a conventional limiting cut-off (regularization) procedure does not yield the known correct results. Since then, many authors have repeatedly returned to the problem of singular potentials, especially to the Calogero problem, investigating its different aspects from different standpoints; see, e.g., [12, 13, 20–30] (the list of references can be significantly extended), including an elucidation of the physical meaning of a formal procedure of s.a. extensions and new parameters involved in terms of regularization and ‘renormalization by square-well counter terms’ [31, 32].

In this paper and the next one, we summarily review all essential mathematical aspects of the one-particle Calogero problem by using a uniform approach based on the theory of s.a. extensions of symmetric differential operators, namely, on a method of specifying s.a. ordinary differential operators associated with s.a. differential expressions by (asymptotic) s.a. boundary conditions and on Krein’s method of guiding functionals for a spectral analysis of ordinary s.a. differential operators.

This paper is organized as follows. To be convinced that a treatment of the Calogero problem requires mathematical accuracy, we begin the exposition with applying the customary physical methods outlined above to this problem, and discuss some QM ‘paradoxes’ inherent in such a ‘naive’ treatment in section 2. In section 3, we construct all possible s.a. operators (s.a. Hamiltonians) associated with the formal differential expression for the Calogero Hamiltonian. A complete spectral analysis of all such s.a. Hamiltonians is given in section 4, where we present their spectra and the corresponding complete sets of (generalized) eigenfunctions. In section 5, we discuss spontaneous scale-symmetry breaking associated with s.a. extensions.

In the next publication, we are going to discuss a new aspect of the problem, the so-called oscillator representation for the Calogero Hamiltonians.

³ To our knowledge, the idea that the mathematical basis for a proper treatment of QM problems with singular potentials is the theory of s.a. extensions of symmetric operators goes back to Berezin and Faddeev, who applied this theory to solving the quantum-mechanical problem with 3-dim. δ -potential [18].

⁴ However, the corresponding analysis was likely to be perceived by physicists of that time as excessively complicated, and in fact, the experience was summarily dismissed.

2. A ‘naive’ treatment of the problem and related paradoxes

As mentioned above, the consideration of this section is on the so-called ‘physical level of rigor’, or, in other words, ‘naive’, so we actually repeat here a negative experience of the first researches.

We start with the formal differential expression, or differential operation,

$$\check{H} = -d_x^2 + \alpha x^{-2}, \quad d_x = d/dx, \quad (2)$$

for the Calogero Hamiltonian, and consider it as a s.a. operator \hat{H} in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+)$ of quantum states for any α , conventionally without any reservations about its domain. We say in advance that the latter is precisely the reason for paradoxes.

In QM, the time evolution governed by a s.a. Hamiltonian \hat{H} is unitary and is defined for all moments of time, although as we have mentioned in the introduction an analog of a ‘fall to the center’ is well known from textbooks in the case of $\alpha < -1/4$: in this case, the spectrum of \hat{H} is unbounded from below. This is argued [15] by considering the singular Calogero potential as a limit of bounded regularized potentials

$$V_{r_0}(x) = \begin{cases} \alpha x^{-2}, & x \geq r_0, \\ \alpha r_0^{-2}, & x < r_0, \end{cases} \quad (3)$$

with $r_0 \rightarrow 0$. However, an attentive reader can see that the spectrum and eigenfunctions have no limit as $r_0 \rightarrow 0$.

We therefore look at the problem in more detail. It is natural to expect that in the case of $\alpha \geq 0$ the spectrum of \hat{H} is nonnegative; the eigenstates are scattering states, and there are no bound states, while in the case of $\alpha < 0$ we expect a bound state of negative energy $E_0 < 0$ in addition to scattering states corresponding to the nonnegative spectrum.

We now turn to some symmetry arguments. It seems evident that the Calogero Hamiltonian has the scale symmetry: under the scale transformations $x \rightarrow x' = lx$, $l > 0$, the operators $\hat{H}_0 = -d_x^2$ and $\hat{V} = \alpha x^{-2}$ transform uniformly and are of the same spatial dimension, $d_{H_0} = d_V = -2$; therefore, the operator \hat{H} also transforms uniformly under scale transformations, and $d_H = -2$. This observation is formalized as follows.

We consider the group of scale transformations $x \rightarrow x' = lx$, $x \in \mathbb{R}_+$, $\forall l > 0$, and its unitary representation in the space $L^2(\mathbb{R}_+)$ of quantum states by unitary operators $\hat{U}(l)$,

$$\hat{U}(l)\psi(x) = l^{-1/2}\psi(l^{-1}x) \quad (4)$$

(the spatial dimension of wavefunctions ψ is $d_\psi = -1/2$ because $|\psi(x)|^2$ is the spatial probability density). The unitarity of $\hat{U}(l)$ is easily verified

$$\|\hat{U}(l)\psi\|^2 = \int_0^{+\infty} dx l^{-1} |\psi(l^{-1}x)|^2 = \int_0^{+\infty} dx |\psi(x)|^2 = \|\psi\|^2,$$

as well as the group law $\hat{U}(l_2)\hat{U}(l_1) = \hat{U}(l_2l_1)$. It is also easily verified that

$$\hat{U}^{-1}(l)\hat{H}\hat{U}(l) = l^{-2}\hat{H} \iff \hat{H}\hat{U}(l) = l^{-2}\hat{U}(l)\hat{H}, \quad (5)$$

or $d_H = -2$.

For completeness, we present an infinitesimal version of scale symmetry. The unitary scale transformations $\hat{U}(l)$ can be represented as

$$\hat{U}(l) = \exp(i \ln l \hat{D}), \quad \hat{D} = ix d_x + i/2 = -(\hat{x}\hat{p} + \hat{p}\hat{x})/2, \quad \hat{p} = -id_x,$$

with \hat{D} being the s.a. generator of scale transformations. The scale-symmetry algebra for the Hamiltonian \hat{H} is $[\hat{D}, \hat{H}] = -2i\hat{H}$.

Let now $\psi_E(x)$ be an eigenfunction of \hat{H} with an eigenvalue E , i.e. $\hat{H}\psi_E(x) = E\psi_E(x)$, then the scale-symmetry operator relation (5) applied to this function yields

$$\hat{H}[\hat{U}(l)\psi_E(x)] = l^{-2}\hat{U}(l)\hat{H}\psi_E(x) = (l^{-2}E)\hat{U}(l)\psi_E(x),$$

which implies that

$$\hat{U}(l)\psi_E(x) = \psi_{l^{-2}E}(x), \quad \forall l > 0,$$

i.e. $\hat{U}(l)\psi_E(x)$ is an eigenfunction of \hat{H} with the eigenvalue $l^{-2}E$. But this implies that the group of scale transformations acts transitively on both the positive and negative parts of the energy spectrum, so that these parts must either be empty or occupy the respective positive and negative semiaxis of the real axis.

This is completely consistent with what we expect for the spectrum of \hat{H} in the case of repulsion, $\alpha > 0$, where $E \geq 0$.

But in the case of attraction, $\alpha < 0$, we meet paradoxes. Indeed, in this case we expect at least one bound state with a negative level, $E_0 < 0$. But if there exists at least one such state, then, according to scale symmetry, there must be a continuous set of normalizable bound states with energies $l^{-2}E_0, \forall l > 0$, and the negative part of the spectrum is the entire negative semiaxis, i.e. a ‘fall to the center’ occurs for all $\alpha < 0$.

This picture is quite unusual and contradictory, because there can be no continuous set of normalizable eigenstates for any s.a. operator in $L^2(\mathbb{R}_+)$: it would contradict the fact that $L^2(\mathbb{R}_+)$ is a separable Hilbert space. Another surprising fact is that the spectrum of the Calogero Hamiltonian is not bounded from below for any $\alpha < 0$, not only for $\alpha < -1/4$.

The situation becomes even more entangled if we try to find boundstates of \hat{H} corresponding to negative energy levels, $E < 0$. The corresponding differential equation for these eigenstates $\psi_E(x) \equiv \psi_k(x)$ is

$$\check{H}\psi_k(x) = -k^2\psi_k(x), \quad k^2 = -E > 0. \tag{6}$$

There are two ‘dangerous’ points for the square integrability of $\psi_k(x)$: the infinity, $x = \infty$, and the origin, $x = 0$, which is a point of singularity of the potential and a boundary simultaneously.

The behavior of a solution $\psi_k(x)$, if it does exist, at infinity where the potential vanishes is evident: $\psi_k(x) \simeq c \exp(-kx), x \rightarrow \infty$. This behavior, which manifests the square integrability of $\psi_k(x)$ at infinity, must be compatible with the local square integrability of $\psi_k(x)$ at the origin. The existence of $\psi_k(x)$ for a given k is thus defined by its asymptotic behavior at the origin, which, because of the singularity, coincides with the asymptotic behavior of the general solution of the homogeneous equation $\check{H}y(x) = 0$ at the origin. The general solution of this equation is

$$y(x) = \begin{cases} x^{1/2}(c_1x^\alpha + c_2x^{-\alpha}), & \alpha \neq -1/4, \\ x^{1/2}(c_1 + c_2 \ln x), & \alpha = -1/4, \end{cases} \tag{7}$$

where

$$\alpha = \sqrt{1/4 + \alpha} = \begin{cases} \sqrt[+]{1/4 + \alpha}, & \alpha \geq -1/4, \\ i\sigma, & \sigma = \sqrt[+]{|1/4 + \alpha|} > 0, \quad \alpha < -1/4. \end{cases} \tag{8}$$

We can see that if $-1/4 \leq \alpha < 0$, we have $\alpha < 1/2$, and $y(x) \rightarrow 0$ as $x \rightarrow 0$, so that $\psi_k(x)$ is certainly square integrable at the origin irrespective of k . The same holds true if $\alpha < -1/4$, in which case $\alpha = i\sigma$ and $y(x) \rightarrow 0$ infinitely oscillating as $x \rightarrow 0$. This implies that $\psi_k(x)$ exists for any $k > 0$, which confirms the previous arguments that the negative ‘discrete’ spectrum is in fact continuous and occupies all the negative real semiaxis.

Furthermore, both functions $x^{1/2 \pm \alpha}$ are also square integrable if $1/2 \leq \alpha < 1$, i.e. if $0 \leq \alpha < 3/4$, so that there is a continuous set of negative energy levels unbounded from

below for $\alpha = 0$ (the case of a free particle) and even for repulsive potentials, $V(x) > 0$. A ‘fall to the center’ for repulsive potentials is quite paradoxical.

We can present the explicit form of $\psi_k(x)$. By the substitution

$$\psi_k(x) = x^{1/2}u_k(kx), \tag{9}$$

we reduce equation (6) to the following equation for the function $u(z) = u_k(kx)$, $z = kx$:

$$u'' + z^{-1}u - (1 + \kappa^2 z^{-2})u = 0, \tag{10}$$

whose solutions are the Bessel functions of imaginary argument. It follows that for $\alpha < 3/4$ and for any $k > 0$ the square-integrable solution of the eigenvalue problem (6) for bound states is given by $\psi_k(x) = x^{1/2}K_\alpha(kx)$, where $K_\alpha(x)$ is the so-called McDonald function.

The final remark is that $\psi_k(x)$ remains square integrable for complex $k = k_1 + ik_2$, $k_1 > 0$, so that the seemingly s.a. \hat{H} has complex eigenvalues.

These inconsistencies, or paradoxes, manifest that something is wrong with QM in the case of singular potentials, as well as in the case of boundaries, or, at least, something is wrong with our previous considerations following the conventional methods. It appears that we have been too ‘naive’ in our considerations; strictly speaking, we have been incorrect, and our arguments have been wrong. The main reason is that almost all operators involved are unbounded, while for unbounded operators, in contrast to bounded operators defined everywhere, the algebraic rules, the notions of self-adjointness, commutativity, and symmetry are nontrivial.

In particular, we actually implicitly adopted that the operator \hat{H} acts (is defined) on the so-called natural domain, which is the set of square-integrable functions ψ satisfying the only conditions that the differential operation \hat{H} is applicable to ψ , and $\hat{H}\psi$ is also square integrable.

As we can see below, this operator with $\alpha < 3/4$ is not s.a.

3. Self-adjoint Calogero Hamiltonians

We now proceed to a more rigorous treatment of the Calogero problem on the semiaxis \mathbb{R}_+ . The first problem to be solved is constructing and suitably specifying all Hamiltonians associated with the differential expression (2) as s.a. operators in the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}_+)$ of QM states; the second problem is a complete spectral analysis of each of the obtained Hamiltonians, and, finally, resolving the paradoxes discussed in the previous section, in particular, the paradox concerning the apparent scale symmetry.

In solving the first problem, we follow [3, 4]; we say in advance that a solution crucially depends on a value of α .

We start with an initial symmetric operator \hat{H} associated with an even s.a. differential expression \check{H} (2) and the operator \hat{H}^+ being the adjoint⁵ of \hat{H} . S.a. Hamiltonians \hat{H}_U are s.a. extensions of the symmetric \hat{H} and simultaneously s.a. restrictions of the adjoint \hat{H}^+ ; the meaning of the subscript U labeling s.a. extensions becomes clear below.

All the above operators form a chain of inclusions $\hat{H} \subset \hat{H}_U \subseteq \hat{H}^+$ and differ only by their domains in $L^2(\mathbb{R}_+)$, while their action on the corresponding domains is given by the same differential expression (2); see⁶ [3].

⁵ From now on, we let the same \hat{H} denote a new operator which differs from the ‘naive Hamiltonian’ \hat{H} in the previous subsection and hope that this will not lead to confusion. In [3], the operators \hat{H} and H^+ were respectively denoted by $\hat{H}^{(0)}$ and H^* . We remind the reader that the self-adjointness of a differential expression is understood in the sense of Lagrange [3].

⁶ Here and elsewhere, we cite our review, where one can find relevant references.

When defining these operators in what follows, we therefore cite only their domains.

The domain D_H of the initial symmetric operator \hat{H} is the space $\mathcal{D}(\mathbb{R}_+)$ of smooth functions with a compact support

$$D_H = \mathcal{D}(\mathbb{R}_+) = \{\varphi(x) : \varphi \in C^\infty(\mathbb{R}_+), \text{supp } \varphi \subseteq [\alpha, \beta] \subset (0, \infty)\}, \quad (11)$$

which is dense in $L^2(\mathbb{R}_+)$. Such a choice of D_H is based on a natural supposition that principal restrictions on functions from D_H must be connected only with peculiarities of the problem, neighborhoods of boundaries and potential singularities. As is known [3], the domain⁷ D_{H^+} of the operator \hat{H}^+ , adjoint to \hat{H} , is the so-called natural domain $D_{*\hat{H}}$ for \hat{H} :

$$D_{H^+} = D_{*\hat{H}} = \{\psi_*(x) : \psi_*, \psi'_* \text{ are a.c. in } \mathbb{R}_+; \psi_*, \hat{H}\psi_* \in L^2(\mathbb{R}_+)\}, \quad (12)$$

where an abbreviation ‘a.c.’ stands for ‘absolutely continuous’.

Self-adjoint Hamiltonians \hat{H}_U are constructed as s.a. restrictions of \hat{H}^+ based on the quadratic asymmetry form Δ_{H^+} which is a measure of the asymmetry of \hat{H}^+ and is defined by⁸

$$\Delta_{H^+}(\psi_*) = (\psi_*, \hat{H}^+\psi_*) - (\hat{H}^+\psi_*, \psi_*), \quad \forall \psi_* \in D_{H^+},$$

see [2]. If $\Delta_{H^+} = 0$, the operator \hat{H}^+ is symmetric and therefore s.a.; the operator \hat{H} is then essentially s.a.; its deficiency indices are $(0, 0)$, and its unique s.a. extension is precisely \hat{H}^+ ; if $\Delta_{H^+} \neq 0$, the deficiency indices of \hat{H} are nonzero, and the domain D_{H_U} of a s.a. operator \hat{H}_U is defined as a maximum subspace of D_{H^+} , where Δ_{H^+} vanishes [2]. If we follow the general theory of s.a. extensions of symmetric operators with equal nonzero deficiency indices (m, m) , these subspaces are determined in terms of the deficient subspaces of the initial symmetric operator \hat{H} and a unitary operator \hat{U} relating them⁹, and there is an m^2 -parameter $U(m)$ -family $\{\hat{H}_U\}$ of s.a. extensions, where $U(m)$ is a unitary group [2].

We now remind the reader of the basic points of a method proposed in [4] for constructing s.a. differential operators \hat{f}_U associated with a general ordinary s.a. differential expression \hat{f} defined on an interval of the real axis in case the associated initial symmetric differential operator \hat{f} with the adjoint \hat{f}^+ has equal deficiency indices.

In our opinion, an advantage of this method is that it avoids evaluating deficiency indices and deficient subspaces of the initial symmetric operator and allows specifying the s.a. operators by explicit s.a. boundary conditions, which is convenient for a subsequent spectral analysis.

For differential operators, the quadratic asymmetry form Δ_{f^+} is represented in terms of quadratic boundary forms [3]. In our case, where both ends of the semiaxis are singular, this representation is

$$\Delta_{H^+}(\psi_*) = [\psi_*, \psi_*](\infty) - [\psi_*, \psi_*](0), \quad [\psi_*, \psi_*](0/\infty) = \lim_{x \rightarrow 0/\infty} [\psi_*, \psi_*](x),$$

where

$$[\psi_*, \psi_*](x) = \overline{\psi'_*(x)}\psi_*(x) - \overline{\psi_*(x)}\psi'_*(x).$$

The quadratic local forms $[\psi_*, \psi_*](0/\infty)$ are the respective left (at the origin) and right (at infinity) boundary forms; these forms do exist (being finite) and are independent.

Each boundary form, if it is nonzero, is a quadratic form in asymptotic boundary coefficients (a.b. coefficients) that are the boundary values of functions $\psi_* \in D_{f^+}$ and

⁷ In [3], the natural domain was denoted simply by D_* . The operator \hat{H}^+ actually coincides with the ‘naive Hamiltonian’ \hat{H} in the previous subsection.

⁸ In [2], this form was denoted by Δ_* .

⁹ To be more precise, \hat{U} is an isometry; the term ‘unitary’ is more conventional for the physical literature.

their derivatives¹⁰ if the respective end (boundary) of the interval is regular or the numerical coefficients in front of the linearly independent leading terms defining the asymptotic behavior of these functions at the respective end and giving a nonzero contribution to the boundary form if the end is singular. Therefore, the asymmetry form Δ_{f^+} is a quadratic in all a.b. coefficients $\{c_k\}_1^{2m}$. Linearly combining the a.b. coefficients into so-called diagonal a.b. coefficients $\{c_{+,k}\}_1^m$ and $\{c_{-,k}\}_1^m$ of the same dimension, we reduce this quadratic form to a diagonal canonical form

$$\Delta_{H^+}(\psi_*) = 2i\kappa \left(\sum_{k=1}^m |c_{+,k}|^2 - \sum_{k=1}^m |c_{-,k}|^2 \right),$$

where κ is a real factor. We note that the inertia indices of the quadratic form coincide with the deficiency indices that are found in passing when finding the a.b. coefficients.

Any s.a. operator \hat{f}_U associated with a given s.a. differential expression \check{f} is uniquely specified by additional boundary conditions at the ends of the interval on the functions $\psi_* \in D_{f^+}$. These boundary conditions are called s.a. boundary conditions; in the presence of singular ends s.a. boundary conditions are of the asymptotic form and are called asymptotic s.a. boundary conditions (a.b. conditions).

(Asymptotic) s.a. boundary conditions are defined by a (fixed) unitary $m \times m$ matrix $U = \|U_{kl}\|, k, l = 1, \dots, m$, that establishes the isometric relation

$$c_{-,k} = \sum_{l=1}^m U_{kl} c_{+,l} \tag{13}$$

between the diagonal a.b. coefficients and thus define a maximum subspace $D_U \subseteq D_{f^+}$, where the quadratic asymmetry form Δ_{f^+} vanishes identically.

The subspace D_U is the domain of a s.a. operator $\hat{f}_U, D_U = D_{f_U}$. In case both ends of the interval are regular, relation (13) is a relation between the boundary values of functions $\psi_U \in D_{f_U}$ and their derivatives, and defines customary boundary conditions. In the case of singular ends, relation (13) prescribes the asymptotic behavior of functions $\psi_U \in D_{f_U}$ at the respective ends; more precisely, the a.b. conditions are formulated as explicit formulas for the leading asymptotic terms of the functions ψ_U at the respective ends.

Conversely, any unitary $m \times m$ matrix U uniquely determines an associated s.a. operator \hat{f}_U by relation (13), so that there exists an m^2 -parameter $U(m)$ family $\{\hat{f}_U\}$ of s.a. operators associated with a given s.a. differential expression \check{f} .

With this ‘instructions’, we return to constructing s.a. Hamiltonians associated with the Calogero differential expression \check{H} (2); it is natural to use the subscript U , or an equivalent one, for the notation of these operators.

The first step consists in evaluating the boundary forms $[\psi_{*,*}, \psi_*](\infty)$ and $[\psi_{*,*}, \psi_*](0)$ in terms of a.b. coefficients.

Because the Calogero potential $V(x)$ (1) vanishes at infinity, we have $\psi_*(x), \psi'_*(x) \rightarrow 0$ as $x \rightarrow \infty$, so that $[\psi_{*,*}, \psi_*](\infty) = 0$. In other words, the infinity turns out to be irrelevant, and the asymmetry form Δ_{H^+} is reduced to the boundary form at the origin and is given by

$$\Delta_{H^+}(\psi_*) = -[\psi_{*,*}, \psi_*](0). \tag{14}$$

Therefore, constructing a s.a. Hamiltonian associated with a s.a. Calogero differential expression \check{H} (2) is reduced to finding a maximum subspace in $D_{*\check{H}}$ where the boundary form at the origin vanishes identically. The physical meaning of the latter condition is clear: because the quadratic local form $[\psi_{*,*}, \psi_*(x)]$ is the probability flux, up to a numerical factor,

¹⁰ In the case of even differential expressions, the derivatives are replaced by so-called quasiderivatives [3].

its vanishing at the origin implies that the probability flux at the origin is zero and a particle does not escape the semiaxis through the left end, together with the zero probability flux at infinity; this implies the unitarity of the time evolution generated by the Hamiltonian that must be s.a.

An evaluation of the boundary form at the origin requires finding a behavior of the wavefunctions $\psi_* \in D_{*\check{H}}$ and their derivatives ψ'_* at the origin. This behavior is conventionally established as follows. We consider the relation $\check{H}\psi_* = \chi$ as a differential equation with respect to the function ψ_* with a given $\chi \in L^2(\mathbb{R}_+)$. If we omit the condition $\psi_* \in D_{*\check{H}}$ for a while, the general solution of this equation and its first derivative allow standard integral representations in terms of the nonhomogeneous term χ and linearly independent solutions $y_1 = x^{1/2+\alpha}$, and y_2 ,

$$y_2 = \begin{cases} x^{1/2-\alpha}, & \alpha \neq -1/4 (\alpha \neq 0), \\ x^{1/2} \ln x, & \alpha = -1/4 (\alpha = 0), \end{cases}$$

of a homogeneous equation, see (7), with the Wronskian

$$\text{Wr}(y_1, y_2) = \begin{cases} -2\alpha, & \alpha \neq -1/4, \\ 1, & \alpha = -1/4, \end{cases}$$

where α is given by (8). These representations are

$$\begin{aligned} \psi_*(x) &= -\frac{x^{1/2}}{2\alpha} \left[x^\alpha \int_a^x d\xi \xi^{1/2-\alpha} \chi - x^{-\alpha} \int_0^x d\xi \xi^{1/2+\alpha} \chi \right] \\ &\quad + c_1 (k_0 x)^{1/2+\alpha} + c_2 (k_0 x)^{1/2-\alpha}, \\ \psi'_*(x) &= -\frac{x^{-1/2}}{2\alpha} \left[(1/2 + \alpha)x^\alpha \int_a^x d\xi \xi^{1/2-\alpha} \chi - (1/2 - \alpha)x^{-\alpha} \int_0^x d\xi \xi^{1/2+\alpha} \chi \right] \\ &\quad + [c_1 (k_0 x)^{1/2+\alpha} + c_2 (k_0 x)^{1/2-\alpha}]', \quad \alpha \neq -1/4 (\alpha \neq 0), \\ \psi_*(x) &= x^{1/2} \left[\int_0^x d\xi \xi^{1/2} \ln(k_0 \xi) \chi - \ln(k_0 x) \int_0^x d\xi \xi^{1/2} \chi \right] \\ &\quad + c_1 x^{1/2} + c_2 x^{1/2} \ln(k_0 x), \\ \psi'_*(x) &= x^{-1/2} \left[\frac{1}{2} \int_0^x d\xi \xi^{1/2} \ln(k_0 \xi) \chi - \left(1 + \frac{1}{2} \ln(k_0 x) \right) \int_0^x d\xi \xi^{1/2} \chi \right] \\ &\quad + [c_1 x^{1/2} + c_2 x^{1/2} \ln(k_0 x)]', \quad \alpha = -1/4, (\alpha = 0), \end{aligned} \tag{15}$$

where k_0 is an arbitrary, but fixed, parameter of the dimension of inverse length introduced by dimensional reasons, $a > 0$ for $\alpha \geq 3/4$, $a = 0$ for $\alpha < 3/4$, and c_1 and c_2 are arbitrary numerical coefficients. We note that $c_{1,2}$ are of the same dimension of one half of the inverse length, the dimension of functions ψ_* , whereas the dimension of functions $\chi = \check{H}\psi_*$ is $5/2$ of the inverse length.

The asymptotic behavior of the integral terms in (15) at the origin is estimated using the Cauchy–Bunyakovskii inequality. For example, if $\alpha > -1/4$ ($\alpha > 0$), the Cauchy–Bunyakovskii inequality yields

$$\left| x^{1/2-\alpha} \int_0^x d\xi \xi^{1/2+\alpha} \chi \right| \leq x^{1/2-\alpha} \left[\int_0^x d\xi \xi^{1+2\alpha} \right]^{1/2} \left[\int_0^x d\xi |\chi|^2 \right]^{1/2},$$

and with the estimates

$$\left[\int_0^x d\xi \xi^{1+2\alpha} \right]^{1/2} = O(x^{1+\alpha}), \quad \left[\int_0^x d\xi |\chi(\xi)|^2 \right]^{1/2} \rightarrow 0 \quad \text{as } x \rightarrow 0$$

(the second estimate follows from the fact that $\chi(\xi) \in L^2(\mathbb{R}_+)$), we find

$$\left| x^{1/2-\alpha} \int_0^x d\xi \xi^{1/2+\alpha} \chi \right| = O(x^{3/2}). \tag{16}$$

The rhs in (16) is overestimated: $O(x^{3/2})$ can be replaced by $o(x^{3/2})$. Similarly, for the integral term $x^{1/2+\alpha} \int_a^x d\xi \xi^{1/2-\alpha} \chi(\xi)$, $a > 0$, we obtain that the estimates $O(x^{3/2})$ and $O(x^{3/2}\sqrt{|\ln x|})$ hold in the respective cases of $\alpha > 3/4$ ($\alpha > 1$), and $\alpha = 3/4$ ($\alpha = 1$), while the estimate $O(x^{3/2})$ holds for the sum of both integral terms in the case of $\alpha = -1/4$.

The asymptotic behavior of the free terms in (15) is evident. If we now restore the condition $\psi_*(x) \in L^2(\mathbb{R}_+)$, we find that $c_2 \neq 0$ in the case of $\alpha \geq 3/4$ ($\alpha \geq 1$) contradicts the condition, because in this case the function $c_2(k_0x)^{1/2-\alpha}$ is not square integrable at the origin unless $c_2 = 0$.

The asymptotic behavior of the derivative ψ'_* at the origin is established quite similarly.

The estimates for the asymptotic behavior of ψ_* and ψ'_* at the origin, i.e. as $x \rightarrow 0$, are finally given by

$$\begin{aligned} \psi_*(x) &= \psi_*^{\text{as}}(x) + \begin{cases} O(x^{3/2}), & \alpha \neq 3/4 (\alpha \neq 1), \\ O(x^{3/2}\sqrt{|\ln x|}), & \alpha = 3/4 (\alpha = 1), \end{cases} \\ \psi'_*(x) &= \psi_*^{\text{as}'}(x) + \begin{cases} O(x^{1/2}), & \alpha \neq 3/4 (\alpha \neq 1), \\ O(x^{1/2}\sqrt{|\ln x|}), & \alpha = 3/4 (\alpha = 1), \end{cases} \end{aligned} \tag{17}$$

where

$$\psi_*^{\text{as}}(x) = \begin{cases} 0, & \alpha \geq 3/4 (\alpha \geq 1), \\ c_1(k_0x)^{1/2+\alpha} + c_2(k_0x)^{1/2-\alpha}, & \begin{cases} -1/4 < \alpha < 3/4 (0 < \alpha < 1) \\ \alpha < -1/4 (\alpha = i\sigma, \sigma > 0), \end{cases} \\ c_1x^{1/2} + c_2x^{1/2} \ln(k_0x), & \alpha = -1/4 (\alpha = 0). \end{cases} \tag{18}$$

The asymptotic estimates (17) and (18) allow a simple calculation of the asymmetry form Δ_{H^+} , given by (14) in terms of a.b. coefficients, and then an explicit formulation of a.b. conditions specifying all s.a. Hamiltonians \hat{H}_U associated with the Calogero differential expression \check{H} (2) via relation (13). It is easy to see that the terms like $O(x^{3/2})$ in ψ_* and $O(x^{1/2})$ in ψ'_* give zero contributions to Δ_{H^+} , while the coefficients c_1 and c_2 are precisely a.b. coefficients. The result crucially depends on the value of α . According to (18), four regions of the values of α are naturally distinguished.

3.1. First region: $\alpha \geq 3/4$ ($\alpha \geq 1$)

For this region of α , the asymmetry form is evidently trivial, $\Delta_{H^+} = 0$. This implies that the operator \hat{H}^+ is symmetric and therefore s.a., which means that the initial symmetric operator \hat{H} has only one s.a. extension¹¹ that is just \hat{H}^+ . In other words, there exists only one s.a. Hamiltonian $\hat{H}_1 = \hat{H}^+$ associated with the s.a. Calogero differential expression \check{H} (2) with $\alpha \geq 3/4$, and it is defined on the natural domain $D_{*\check{H}}$ (12),

$$D_{H_1} = D_{*\check{H}}. \tag{19}$$

The functions $\psi_*(x) \in D_{*\check{H}}$ vanish at the origin, $\psi_*(x) \rightarrow 0$ as $x \rightarrow 0$, which is quite natural from the physical standpoint for a strongly repulsive Calogero potential (1) with $\alpha \geq 3/4$ ($\alpha > 1$). The standard heuristic physical arguments when finding eigenfunctions in the Calogero potential are based on a realistic hypothesis that the behavior of the eigenfunctions at the origin is defined by the homogeneous equation $\check{H}\psi_E = 0$. Its solution, see (7), yields

¹¹ In passing, we find that \hat{H} has zero deficiency indices and is therefore essentially s.a..

$\psi_E(x) \simeq c_1 x^{1/2+\kappa} + c_2 x^{1/2-\kappa}$, $x \rightarrow 0$. The second term on the rhs must be omitted, because it is not square integrable at the origin for $\kappa > 1$ unless $c_2 = 0$, and $\psi_E(x)$ must vanish as $x \rightarrow 0$. But we can represent this vanishing as $x \rightarrow 0$ more precisely, see (17) and (18),

$$\begin{aligned} \psi_*(x) &= O(x^{3/2}), & \psi'_*(x) &= O(x^{1/2}), & \alpha &> 3/4, \\ \psi_*(x) &= O(x^{3/2}\sqrt{|\ln x|}), & \psi'_*(x) &= O(x^{1/2}\sqrt{|\ln x|}), & \alpha &= 3/4; \end{aligned} \tag{20}$$

this estimate also holds true for the eigenfunctions ψ_E .

3.2. Second region: $-1/4 < \alpha < 3/4$ ($0 < \kappa < 1$)

In this region, the asymmetry form Δ_{H^+} is

$$\Delta_{H^+}(\psi_*) = -2k_0\kappa(\overline{c_1}c_2 - \overline{c_2}c_1) = ik_0\kappa(|c_+|^2 - |c_-|^2), \quad c_{\pm} = c_1 \pm ic_2,$$

where c_{\pm} are the diagonal a.b. coefficients¹². In such a case, the matrix U in (13) is reduced to a complex number of unit module, an element of the group $U(1)$, that is a circle, and relation (13) becomes $c_- = e^{i\vartheta} c_+$, $0 \leq \vartheta \leq 2\pi$, $0 \sim 2\pi$, or, equivalently¹³,

$$c_2 = \lambda c_1, \quad \lambda = -\tan \vartheta/2 \in \overline{\mathbb{R}}, \tag{21}$$

and $|\lambda| = \infty$ means that $c_1 = 0$, c_2 is arbitrary.

Relation (21) with any fixed λ defines a maximum subspace $D_{2,\lambda} \subset D_{*\check{H}}$ where the asymmetry form Δ_{H^+} vanishes identically. The subspace $D_{2,\lambda}$ is the domain of a s.a. operator $\hat{H}_{2,\lambda}$, $D_{2,\lambda} = D_{H_{2,\lambda}}$, specified by a.b. conditions at the origin

$$\begin{aligned} \psi_{2,\lambda}(x) &= \psi_{2,\lambda}^{as}(x) + O(x^{3/2}), & \psi'_{2,\lambda}(x) &= \psi_{2,\lambda}^{as'}(x) + O(x^{1/2}) & \text{as } x \rightarrow 0, \\ \psi_{2,\lambda}^{as}(x) &= \begin{cases} c[(k_0x)^{1/2+\kappa} + \lambda(k_0x)^{1/2-\kappa}], & |\lambda| < \infty, \\ c(k_0x)^{1/2-\kappa}, & |\lambda| = \infty. \end{cases} \end{aligned} \tag{22}$$

We thus obtain that in the case of $-1/4 < \alpha < 3/4$; constructing a s.a. Hamiltonian associated with the s.a. Calogero differential expression \check{H} (2) is nonunique: there exists a one-parameter $U(1)$ family $\{\hat{H}_{2,\lambda}, |\lambda| \leq \infty\}$ of s.a. Hamiltonians, and their domains $D_{H_{2,\lambda}}$ are given by

$$D_{H_{2,\lambda}} = \{\psi_{\lambda} : \psi_{\lambda} \in D_{*\check{H}}; \psi_{\lambda}, \psi'_{\lambda} \text{ satisfy (22)}\}. \tag{23}$$

A concrete choice of λ , and therefore that of a s.a. Hamiltonian, requires additional arguments.

This problem is well known in physics and is solved, based on additional physical considerations [15]. According to the above physical arguments, an asymptotic behavior of eigenfunctions at the origin is given by the previous formula $\psi_E(x) \simeq c_1 x^{1/2+\kappa} + c_2 x^{1/2-\kappa}$, $x \rightarrow 0$, but now $0 < \kappa < 1$; both terms are therefore square integrable at the origin, and there arises an unexpected uncertainty in the choice of boundary conditions at the origin and that of scattering states. To avoid this difficulty, it is proposed to consider the regularized cut-off potential (3) with the standard boundary conditions $\psi_E(0) = 0$ and the subsequent limit $r_0 \rightarrow 0$. This limiting procedure yields $c_2 = 0$, or the choice $\lambda = 0$ in the a.b. conditions (22). Under this choice, we have customary zero boundary conditions at the origin for wavefunctions for all α in the interval $(-1/4, 3/4)$, while for their derivatives we have zero boundary conditions if $0 < \alpha < 3/4$ (repulsion) and specific singularities at the origin if $-1/4 < \alpha < 0$ (attraction). With $\lambda \neq 0$, we have specific singularities at the origin for wavefunctions if $0 < \alpha < 3/4$ and zero boundary conditions if $-1/4 < \alpha < 0$, while for the

¹² In passing, we find that the deficiency indices of the initial symmetric operator \hat{H} with α such that $-1/4 < \alpha < 3/4$ are $(1, 1)$ and thus there must exist a one-parameter family of s.a. extensions of \hat{H} .

¹³ The symbol $\overline{\mathbb{R}}$ here denotes the compactified real axis where $-\infty$ and $+\infty$ are identified: $\overline{\mathbb{R}} = \{\lambda : -\infty \leq \lambda \leq +\infty, -\infty \sim +\infty\}$; $\overline{\mathbb{R}}$ is homeomorphic to a circle.

derivatives of wavefunctions we have specific singularities for all α . We make a remark on a possible physical meaning of s.a. Hamiltonians with $\lambda \neq 0$, which cannot be obtained by the above regularization procedure. If we treat a one-dimensional Calogero Hamiltonian as a radial Hamiltonian in the three- or two-dimensional Calogero problem where $x = r$, the additional terms with the factor λ in a.b. conditions can be treated as a manifestation of additional singular terms of zero radius in the potential; we call them δ -like terms. There are different arguments in favor of this suggestion. First, such potentials are not grasped by an initial symmetric operator, whose domain is a set of functions vanishing at the origin. Second, even in the case of free motion, $\alpha = 0$, the a.b. conditions with $\lambda \neq 0$ are admissible, and it was first shown in [18] that a local potential $V \sim \delta^{(3)}(\mathbf{r})$ can properly be treated in terms of s.a. extensions, with $\lambda \neq 0$, of the Laplacian Δ_3 , initially defined as a symmetric operator on functions vanishing at the origin. An additional heuristic argument is that three-dimensional radial functions square integrable with the measure $r^2 dr$ differ from our one-dimensional functions by the factor $1/r$; therefore, if $\lambda \neq 0$, the asymptotic behavior of the s -wave radial function is given by λ/r , and we formally obtain $\Delta_3 \lambda/r \sim \lambda \delta^{(3)}(\mathbf{r})$. Finally, a credible speculation is that δ -like terms can be reproduced, and therefore any s.a. Hamiltonian $\hat{H}_{2,\lambda}$ can be obtained by means of a limiting procedure $r_0 \rightarrow 0$ if we start with a more sophisticated regularized potential where the cut-off potential (3) is supplemented by an attractive or repulsive potential of the same radius (a square well or a core, respectively) whose strength is appropriately fitted to r_0 in terms of a certain finite λ that survives in a.b. conditions in the limit $r_0 \rightarrow 0$. A verification of this hypothesis is an interesting problem for a further study.

3.3. Third region: the point $\alpha = -1/4$ ($\alpha = 0$)

The consideration of this value of α and the result are completely similar to those in the previous subsection.

The expression for the asymmetry form is identical to that of the previous subsection¹⁴:

$$\Delta_{H^+} = -k_0(\bar{c}_1 c_2 - \bar{c}_2 c_1) = \frac{i}{2} k_0(|c_+|^2 - |c_-|^2), \quad c_{\pm} = c_1 \pm i c_2,$$

the relation between c_- and c_+ under which Δ_{H^+} vanishes identically is $c_- = e^{i\vartheta} c_+$, $0 \leq \vartheta \leq 2\pi$, $0 \sim 2\pi$, or, equivalently,

$$c_1 = \lambda c_2, \quad \lambda = -\cot \vartheta/2, \quad -\infty \leq \lambda \leq \infty, \quad -\infty \sim \infty. \quad (24)$$

Relation (24) with any fixed λ defines a.b. conditions at the origin,

$$\begin{aligned} \psi_{3,\lambda}(x) &= \psi_{3,\lambda}^{\text{as}}(x) + O(x^{3/2}), & \psi'_{3,\lambda}(x) &= \psi_{3,\lambda}^{\text{as}'}(x) + O(x^{1/2}) & \text{as } x \rightarrow 0, \\ \psi_{3,\lambda}^{\text{as}}(x) &= \begin{cases} c[\lambda x^{1/2} + x^{1/2} \ln(k_0 x)], & |\lambda| < \infty, \\ c x^{1/2}, & |\lambda| = \infty, \end{cases} \end{aligned} \quad (25)$$

specifying a s.a. operator $\hat{H}_{3,\lambda}$.

The final conclusion is similar to that of the previous subsection: there exists a one-parameter $U(1)$ family $\{\hat{H}_{3,\lambda}, |\lambda| \leq \infty\}$ of s.a. Calogero Hamiltonians associated with the s.a. Calogero differential expression \check{H} (2) with $\alpha = -1/4$, and their domains $D_{H_{3,\lambda}}$ are given by

$$D_{H_{3,\lambda}} = \{\psi_{3,\lambda} : \psi_{3,\lambda} \in D_{*\check{H}}; \psi_{3,\lambda}, \psi'_{3,\lambda} \text{ satisfy (25)}\}. \quad (26)$$

We can also add a remark, similar to that in the end of the previous subsection, on a concrete choice of λ and on a possible physical meaning of the latter. The case $\alpha = -1/4$ corresponds

¹⁴ We can repeat the remark in footnote 12 for this value of α .

to a free motion of a particle in the two-dimensional space, and the two-dimensional radial functions square integrable with the measure $r dr$ differ from our one-dimensional functions by the factor $r^{-1/2}$; therefore, if $|\lambda| \neq \infty$, the asymptotic behavior of the s -wave radial function is given by $\lambda^{-1} \ln(k_0 r)$, and we formally obtain $\Delta_2 \lambda^{-1} \ln(k_0 r) \sim \lambda^{-1} \delta^{(2)}(\mathbf{r})$.

3.4. Fourth region: $\alpha < -1/4$ ($\alpha = i\sigma, \sigma > 0$)

A consideration for these values of α is a copy of those in the previous two subsections¹⁵.

The asymmetry form is a canonical diagonal form from the very beginning:

$$\Delta_{H^*}(\psi_*) = i2k_0\sigma(|c_1|^2 - |c_2|^2).$$

It follows that there exists a one-parameter $U(1)$ family $\{\hat{H}_{4,\theta}, 0 \leq \theta \leq \pi, 0 \sim \pi\}$ of s.a. Hamiltonians associated with the s.a. Calogero differential expression \check{H} (2) with $\alpha < -1/4$ and specified by a.b. conditions at the origin,

$$\begin{aligned} \psi_{4,\theta}(x) &= \psi_{\theta}^{\text{as}}(x) + O(x^{3/2}), & \psi'_{4,\theta}(x) &= \psi_{\theta}^{\text{as}'}(x) + O(x^{1/2}), & x \rightarrow 0, \\ \psi_{4,\theta}^{\text{as}}(x) &= cx^{1/2}[e^{i\theta}(k_0x)^{i\sigma} + e^{-i\theta}(k_0x)^{-i\sigma}]. \end{aligned} \tag{27}$$

The domain of each of the Hamiltonian $\hat{H}_{4,\theta}$ is given by

$$D_{H_{4,\theta}} = \{\psi_{4,\theta} : \psi_{4,\theta} \in D_{*\check{H}}; \psi_{4,\theta}, \psi'_{4,\theta} \text{ satisfy (27)}\}. \tag{28}$$

The leading terms in a.b. conditions (27) can be written as follows:

$$\psi_{4,\theta}^{\text{as}}(x) = cx^{1/2} \cos(\sigma \ln k_0x + \theta), \tag{29}$$

and 2θ can be interpreted as the phase of the scattered wave at the origin. This form of a.b. conditions was first proposed in [17].

It is interesting to compare the result with conventional physical considerations. The standard arguments concerning the asymptotic behavior of eigenfunctions ψ_E at the origin yield $\psi_E(x) \simeq x^{1/2}(c_1x^{i\sigma} + c_2x^{-i\sigma}), x \rightarrow 0$. In contrast to the previous cases, both terms are of the same infinitely oscillating behavior, which does not allow definitely fixing c_1 and c_2 , more precisely, fixing the ratio c_2/c_1 , and thus constructing scattering states.

An attempt to fix this ratio via the limiting procedure $r_0 \rightarrow 0$ starting with the cut-off potential (3) fails: c_2/c_1 has no limit as $r_0 \rightarrow 0$, and there is no limit for eigenvalues and eigenfunctions [15, 19]. However, as stated in [32], superimposing the cut-off potential (3) with a square-well attractive potential of the same radius, and thus changing the potential $V_{r_0}(x)$ to the potential $V_s(x) = -\alpha_s(r_0)r_0^{-2}\theta(r_0 - x) - \alpha r^{-2}\theta(x - r_0)$, where θ stands for the symbol of the known step function, allows for obtaining a.b. conditions (29) in the limit of zero radius r_0 under an appropriate choice of the coupling constant $\alpha_s(r_0)$.

4. Spectral analysis

4.1. Preliminary

We now turn to a spectral analysis of Hamiltonians consisting of the above four families in accordance with different values of the coupling constant α . This includes finding the spectrum and (generalized) eigenfunctions for each Hamiltonian and deriving formulas for the respective eigenfunction expansions of the arbitrary square-integrable function. The short name for these formulas in mathematics is ‘inversion formulas’.

¹⁵ We can repeat the remark in footnote 12 for this value of α .

In what follows, we use this term. In solving the spectral problem, we follow Krein’s method of guiding functionals where the spectrum and eigenfunctions emerge in the process of deriving inversion formulas [6, 7]. For differential operators of second order, we generally need two guiding functionals. But in the case where the spectrum is expected to be simple, as in our case, it suffices to have only one, the so-called simple guiding functional. We remind the reader of the basics of Krein’s method in this case¹⁶ as applied to our problem.

Let \hat{H} be a s.a. Hamiltonian associated with the differential expression \check{H} (2); by \hat{H} , we mean any operator in the above four families. Krein’s method for a spectral analysis of \hat{H} rests on using certain solutions of the homogeneous differential equation

$$(\check{H} - W)u(x; W) = 0, \tag{30}$$

where $W = \text{Re}W + i\text{Im}W = E + i\text{Im}W$ is an arbitrary complex number; we can say that $u(x; W)$ is ‘an eigenfunction of \check{H} with complex energy W ’.

Let $u(x; W)$ be a function with the following properties:

- (i) $u(x; W)$ is a solution of the homogeneous equation (30),
- (ii) $u(x; W)$ is real entire in W , i.e. it is an entire function of W for each fixed x that is real for a real W : $u(x; E) = \overline{u(x; \bar{E})}$,
- (iii) $u(x; W)$ satisfies a.b. conditions specifying the Hamiltonian \hat{H} under consideration.

Such a function does exist (see below).

Let \mathbb{D} be a space of functions ξ belonging to the domain of \hat{H} and vanishing for $x > b > 0$, where b may be different for each ξ , i.e. $\mathbb{D} = D_H \cap D_r(\mathbb{R}_+)$, where $D_r(\mathbb{R}_+)$ is the space of functions in \mathbb{R}_+ with a support bounded from the right. The space \mathbb{D} is dense in $L^2(\mathbb{R}_+)$, $\overline{\mathbb{D}} = L^2(\mathbb{R}_+)$. The linear functional $\Phi(\xi; W)$ defined in a space \mathbb{D} and given by

$$\Phi(\xi; W) = \int_0^\infty u(y; W)\xi(y) dy, \quad \forall \xi \in \mathbb{D}, \tag{·}$$

is called a guiding functional; the integration on the rhs of (·) actually goes up to some finite b .

It is evident that $\Phi(\xi; W)$ is an entire function of W for each fixed ξ and obeys the property $\Phi(\check{H}\xi; W) = W\Phi(\xi; W)$, which follows from the Lagrange identity for the functions u and ξ satisfying the same a.b. conditions at the origin. Let the functional Φ also obey the property

$$\Phi(\xi_0; E_0) = 0, \quad \xi_0 \in \mathbb{D}_H \implies \exists \psi_0 \in \mathbb{D}_H, \quad (\check{H} - E_0)\psi_0 = \xi_0, \tag{31}$$

then we call Φ a simple guiding functional.

Let the functional Φ be simple. Then there hold the following statements.

- (1) The spectrum of the Hamiltonian \hat{H} , $\text{spec } \hat{H}$, is simple.
- (2) There exists a spectral function $\sigma(E)$ for the Hamiltonian \hat{H} such that $\text{spec } \hat{H}$ is the set of its growth points. The spectral function defines the Hilbert space L^2_σ with the measure $d\sigma(E)$.
- (3) For any function $\psi \in L^2(\mathbb{R}_+)$, the inversion formulas

$$\psi(x) = \int \varphi(E)u(x; E) d\sigma(E), \quad \varphi(E) = \int_0^\infty u(x; E)\psi(x) dx \in L^2_\sigma \tag{32}$$

hold true, together with the Parseval equality

$$\int_0^\infty |\psi(x)|^2 dx = \int |\varphi(E)|^2 d\sigma(E). \tag{33}$$

¹⁶ These were outlined in [5].

The integrals on the rhs of (32) converge in the respective $L^2(\mathbb{R}_+)$ and L^2_σ . The integration over E on the rhs of (32) and (33) goes over $\text{spec } \hat{H}$, so that we can set $\varphi(E) = 0$ if $E \notin \text{spec } \hat{H}$, i.e. for all constancy points of $\sigma(E)$, and $u(x; E)$ for such E do not enter the inversion formulas.

- (4) Let the spectral function be the sum of a jump function $\sigma_{\text{jmp}}(E)$ and of an absolutely continuous function $\sigma_{\text{a.c.}}(E)$, $\sigma(E) = \sigma_{\text{jmp}}(E) + \sigma_{\text{a.c.}}(E)$, as in our case¹⁷. Then $d\sigma(E)$ can be represented as $d\sigma(E) = \sigma'(E) dE$, where the derivative $\sigma'(E)$, the so-called spectral density, is understood in the distribution sense and is given by

$$\sigma'(E) = \sum_n \rho_n \delta(E - E_n) + \rho_c(E), \quad \rho_n > 0, \quad \rho_c(E) \geq 0.$$

The set $\{E_n\}$ can be empty; if not, the real numbers $E_n, \{n\} \subset \mathbb{Z}$, are the energy eigenvalues for the Hamiltonian \hat{H} corresponding to bound states. The set $\{E_n\}$ of bound-state energies is the discrete spectrum (or the discrete part of the spectrum) of \hat{H} , while the set $\text{supp } \rho_c(E)$, is the continuous spectrum (or the continuous part of the spectrum) of \hat{H} , and the whole spectrum of \hat{H} is the union of these sets $\text{spec } \hat{H} = \{E_n\} \cup \text{supp } \rho_c(E)$. Accordingly, the functions $u(x; E_n)$ are normalizable eigenfunctions of bound states of \hat{H} , while $u(x; E), E \in \text{supp } \rho_c(E)$, are (generalized) eigenfunctions of the continuous spectrum of \hat{H} .

If we introduce normalized eigenfunctions $u_{\text{nr}}(x; E)$ by

$$u_{\text{nr}}(x; E) = \begin{cases} u_n(x) = \sqrt{\rho_n} u(x; E_n), & E = E_n, \\ u_E(x) = \sqrt{\rho(E)} u(x; E), & E \in \text{supp } \rho_c(E), \end{cases}$$

then the inversion formulas (32) and the Parseval equality (33) become

$$\psi(x) = \sum_n \varphi_n u_n(x) + \int_{\text{supp } \rho_c(E)} \varphi(E) u_E(x) dE, \tag{34}$$

$$\varphi(E) = \int_0^\infty u_E(x) \psi(x) dx, \quad \varphi_n = \int_0^\infty u_n(x) \psi(x) dx, \tag{35}$$

$$\int_0^\infty |\psi(x)|^2 dx = \sum_n |\varphi_n|^2 + \int_{\text{supp } \rho_c(E)} |\varphi(E)|^2 dE. \tag{36}$$

We note that the inversion formulas and the Parseval equality do not change (are invariant) under the change of the sign of any of the bound-state eigenfunctions (and of the continuous-spectrum eigenfunctions), which can sometimes be convenient.

In what follows, when completing the spectral analysis of each family of s.a. Calogero Hamiltonians, we do not present the respective inversion formulas explicitly, but restrict ourselves to an assertion of the type ‘the above given set of eigenfunctions form a complete orthonormalized system of eigenfunctions for the Hamiltonian under consideration’; by such an assertion we mean that there hold inversion formulas and a Parseval equality of forms (34)–(36) in terms of the corresponding normalized eigenfunctions. Using this terminology, we follow the physical tradition. Formulas (34)–(36) are of customary form for physicists. From the physical standpoint, these formulas testify that the eigenfunctions form a complete orthogonal system in the sense that they satisfy the respective completeness and orthonormality relations

$$\sum_n u_n(x) u_n(x') + \int_{c.\text{spec } \hat{H}} u_E(x) u_E(x') dE = \delta(x - x'), \tag{37}$$

¹⁷ The spectral function does not contain so-called singular terms.

$$\int_0^\infty u_n(x)u_{n'}(x) dx = \delta_{nn'}, \quad \int_0^\infty u_n(x)u_E(x) dx = 0, \tag{38}$$

$$\int_0^\infty u_E(x)u_{E'}(x) dx = \delta(E - E'), \quad E, E' \in \text{c.spec } \hat{H},$$

where $\text{c.spec } \hat{H}$ denotes the continuous spectrum of \hat{H} ; here, $\text{c.spec } \hat{H} = \text{supp } \rho_c(E)$. In physical texts on QM, the main effort is usually made to establish precisely these relations, and the last relation in (38) is conventionally called ‘the normalization of the continuous-spectrum eigenfunctions to the δ function’. It is needless to say that, from the mathematical standpoint, the relations in (37) and (38) containing δ functions are at most heuristic.

- (5) The spectral function $\sigma(E)$ is evaluated via Green’s function $G(x, y; W)$ of the Hamiltonian \hat{H} that is the integral kernel of the resolvent $\hat{R}(W) = (\hat{H} - W)^{-1}$, i.e. a kernel of the integral representation $\psi(x) = \int_0^\infty G(x, y; W)\chi(y) dy$ for a unique solution $\psi \in D_H$ of the nonhomogeneous equation $(\hat{H} - W)\psi = \chi$ with an arbitrary square integrable rhs χ , $\chi \in L^2(\mathbb{R}_+)$. It suffices to consider W in the upper half-plane, $\text{Im}W > 0$. Namely, if we introduce the function

$$M(c; W) = G(c - 0, c + 0; W), \tag{39}$$

where c is an arbitrary inner point of \mathbb{R}_+ , $c \in (0, \infty)$, then the spectral density $\sigma'(E)$ is determined by the relation

$$[u(c; E)]^2\sigma'(E) = \pi^{-1} \text{Im } M(c; E + i0). \tag{40}$$

It remains to find a convenient representation for Green’s function that allows finding the spectral density.

Let $v(x; W)$ be a solution of the homogeneous equation (30) that is linearly independent of the solution $u(x; W)$ and exponentially decreases at infinity (such a solution does exist). Then Green’s function is given by

$$G(x, y; W) = \omega^{-1}(W) \begin{cases} v(x; W)u(y; W), & x > y, \\ u(x; W)v(y; W), & x < y, \end{cases} \tag{41}$$

where $\omega(W) = -\text{Wr}(u, v)$, and therefore, the function $M(c; W)$ (39) is given by

$$M(c; W) = \omega^{-1}(W)u(c; W)v(c; W). \tag{42}$$

Formulas (40)–(42) suffice for evaluating the spectral function. However, from the calculation standpoint, the following modification may appear to be more suitable. Let $\tilde{u}(x; W)$ be a solution of the homogeneous equation (30) that is real entire, as well as the solution $u(x; W)$, but is linearly independent of $u(x; W)$, so that their Wronskian is $\text{Wr}(u, \tilde{u}) = -\tilde{\omega}(W) \neq 0$, and the function $\tilde{\omega}(W)$ is real entire. The functions u and \tilde{u} form a fundamental set of solutions of the homogeneous equation (30); therefore, the function v allows the representation

$$v(x, W) = c_1(W)u(x; W) + c_2(W)\tilde{u}(x; W),$$

$$c_1(W) = -\frac{\text{Wr}(v, \tilde{u})}{\tilde{\omega}(W)}, \quad c_2(W) = \frac{\omega(W)}{\tilde{\omega}(W)}, \tag{43}$$

$$c_{1,2}(W) \neq 0 \quad \text{for } \text{Im } W > 0.$$

We note that the function v is defined up to a nonzero factor, and we can make a change $v \rightarrow v/c_1$; then representation (43) becomes

$$v(x, W) = u(x; W) + \frac{\omega(W)}{\tilde{\omega}(W)}\tilde{u}(x; W). \tag{44}$$

With such a choice of v and (41) and (42) taken into account, equation (40) becomes

$$\sigma'(E) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \text{Im} \omega^{-1}(E + i\varepsilon) \tag{45}$$

because $u(c; E + i0) = u(c; E)$, $\tilde{u}(c; E + i0) = \tilde{u}(c; E)$, and $\tilde{\omega}(E + i0) = \tilde{\omega}(E)$ are real.

We should clarify formula (45). The function $\omega(E) = \omega(E + i0) = \lim_{\varepsilon \rightarrow +0} \omega(E + i\varepsilon)$ can have simple isolated zeroes. If $\omega(E) \neq 0$, then $\lim_{\varepsilon \rightarrow +0} \text{Im} \omega^{-1}(E + i\varepsilon) = \text{Im} \omega^{-1}(E)$, and therefore,

$$\sigma'(E) = \pi^{-1} \text{Im} \omega^{-1}(E) \quad \text{if } \omega(E) \neq 0. \tag{46}$$

Let E_0 be a simple isolated zero of the function $\omega(E)$, $\omega(E_0) = 0$. The function $\omega^{-1}(W)$ allows the representation

$$\omega^{-1}(W) = [\omega'(E_0)(W - E_0)]^{-1} + \phi(W), \quad \omega'(E_0) < 0,$$

where the function $\phi(W) = \omega^{-1}(W) - [\omega'(E_0)(W - E_0)]^{-1}$ is nonsingular in some neighborhood of the point E_0 . Using the known formula

$$\lim_{\varepsilon \rightarrow +0} \text{Im}(E - E_0 + i\varepsilon)^{-1} = -\pi \delta(E - E_0),$$

we then obtain that in this neighborhood there holds the representation

$$\lim_{\varepsilon \rightarrow +0} \text{Im} \omega^{-1}(E + i\varepsilon) = -\frac{\pi}{\omega'(E_0)} \delta(E - E_0) + \text{Im} \phi(E),$$

where

$$\text{Im} \phi(E) = \text{Im} \omega^{-1}(E), \quad E \neq E_0, \quad \text{Im} \phi(E_0) = \lim_{E \rightarrow E_0} \text{Im} \phi(E),$$

in particular, $\text{Im} \phi(E) = 0$ if $\omega(E)$ is real, and we find that

$$\sigma'(E) = -(\omega'(E_0))^{-1} \delta(E - E_0) \quad \text{if } \omega(E_0) = 0, \tag{47}$$

and $\text{Im} \omega(E) = 0$ in a neighborhood of E_0 . After this, we proceed with a direct spectral analysis of the s.a. Calogero Hamiltonians, sequentially from the first family to the fourth one in accordance with the different regions of values of the coupling constant α . In each region, the spectral analysis and its result have some specific features. We also remember that in each case we must verify that a chosen guiding functional is simple, i.e. that property (31) for a given guiding functional holds true. Because a simple substitution reduces the homogeneous equation (30) to the Bessel equation, see (6)–(10), the above functions u , v and \tilde{u} are different Bessel functions up to the factor $x^{1/2}$; we hope that the cited properties of Bessel functions are well known or can be easily taken out of handbooks on special functions. The necessary Wronskians can be evaluated using the asymptotic expansions of the corresponding functions at the origin. As a rule, we label all the functions involved by indices indicating a family and an extension parameter, as well as the corresponding Hamiltonians.

4.2. First region: $\alpha \geq 3/4$ ($\chi \geq 1$)

In this region, there exists only one s.a. Calogero Hamiltonian \hat{H}_1 defined on the natural domain $D_{H_1} = D_{*\hat{H}}$.

For the functions u and v , the above-described special solutions of the homogeneous equation (30) are taken as the respective

$$u_1(x; W) = (\beta/2k_0)^{-\chi} x^{1/2} J_\chi(\beta x), \tag{48}$$

$$v_1(x; W) = (\beta/2k_0)^\chi x^{1/2} H_\chi^{(1)}(\beta x), \tag{49}$$

where J_x is the Bessel function, $H_x^{(1)}$ is the Hankel function, and we set $W = |W| \exp i\varphi$, $0 < \varphi < \pi$, $\beta = \sqrt{W} = \sqrt{|W|} \exp i\varphi/2$, $\text{Im } \beta > 0$, while k_0 is a (fixed) parameter of a dimension of the inverse length introduced by dimensional reasons. The asymptotic behavior of these functions at the origin, as $x \rightarrow 0$, is given by

$$u_1(x; W) = \frac{k_0^{-1/2}}{\Gamma(1 + \varkappa)} (k_0 x)^{1/2 + \varkappa} [1 + O(x^2)], \tag{50}$$

$$v_1(x; W) = -i \frac{k_0^{-1/2} \Gamma(\varkappa)}{\pi} (k_0 x)^{1/2 - \varkappa} [1 + O(x^2)], \tag{51}$$

whence it follows that $\omega_1(W) = -\text{Wr}(u_1, v_1) = -2i/\pi$. Their asymptotic behavior at infinity, as $x \rightarrow \infty$, is given by

$$\begin{aligned} u_1(x; W) &= \frac{1}{2\sqrt{\pi k_0}} (2k_0/\beta)^{1/2 + \varkappa} e^{-i(\beta x - \varkappa\pi/2 - \pi/4)} [1 + O(x^{-1})] \rightarrow \infty, \\ v_1(x; W) &= \frac{2\sqrt{k_0}}{\beta\sqrt{\pi}} (\beta/2k_0)^{1/2 + \varkappa} e^{i(\beta x - \varkappa\pi/2 - \pi/4)} [1 + O(x^{-1})] \rightarrow 0. \end{aligned} \tag{52}$$

It is easy to see from (48) and (50) that the function $u_1(x; W)$ is real entire in W and obeys the required a.b. conditions (20). The guiding functional is

$$\Phi(\xi; W) = \int_0^\infty u_1(y; W) \xi(y) dy, \quad \xi \in \mathbb{D} = D_{*\check{H}}(\mathbb{R}_+) \cap D_r(\mathbb{R}_+). \tag{53}$$

We check that Φ meets property (31). Let

$$\Phi(\xi_0; E_0) = 0, \quad \xi_0 \in \mathbb{D}. \tag{54}$$

Because $\xi_0 \in \mathbb{D}$, its support is bounded, $\text{supp } \xi_0 \subseteq [0, b]$ with some $b < \infty$, and (54) is equivalent to

$$\int_0^b u_1(y; E_0) \xi_0(y) dy = 0. \tag{55}$$

We consider the function ψ_0 given by

$$\psi_0(x) = \frac{i\pi}{2} \left[v_1(x; E_0) \int_0^x u_1(y; E_0) \xi_0(y) dy + u_1(x; E_0) \int_x^b v_1(y; E_0) \xi_0(y) dy \right]. \tag{56}$$

This function evidently satisfies the equation $(\check{H} - E_0) \psi_0 = \xi_0$, and

$$\psi'_0(x) = \frac{i\pi}{2} \left[v'_1(x; E_0) \int_0^x u_1(y; E_0) \xi_0(y) dy + u'_1(x; E_0) \int_x^b v_1(y; E_0) \xi_0(y) dy \right]. \tag{57}$$

If we take account of (50) and (51), the asymptotic behavior of ψ_0 and ψ'_0 at the origin is simply estimated by means of the Cauchy–Bunyakovskii inequality for the integrals in (56) and (50), to yield $\psi(x) = O(x^{3/2})$ and $\psi'(x) = O(x^{1/2})$ as $x \rightarrow 0$.

On the other hand, for sufficiently large x , we have

$$\psi_0(x) = \frac{i\pi}{2} v_1(x; E_0) \int_0^b u_1(y; E_0) \xi_0(y) dy = 0, \quad x > b,$$

in view of (55), i.e. $\text{supp } \psi_0 \subseteq [0, b]$. This means that $\psi_0 \in \mathbb{D}$, and the guiding functional (53) is therefore simple.

The function M (42) in this region is

$$M(c; W) = \frac{i\pi}{2} u_1(c; W) v_1(c; W) = \frac{i\pi}{2} c J_x(\beta c) H_x^{(1)}(\beta c).$$

Let $E = -\tau^2 < 0$, $\tau > 0$, $\beta = e^{i\pi/2}\tau$. Using the representations

$$J_\nu(e^{i\pi/2}\tau x) = e^{\frac{i\pi\nu}{2}} I_\nu(\tau x), \quad H_\nu^{(1)}(e^{i\pi/2}\tau x) = e^{-\frac{i\pi\nu}{2}} \frac{2}{i\pi} K_\nu(\tau x), \quad (58)$$

where I_ν is the modified Bessel function (of the first kind), and K_ν is the McDonald function¹⁸, which are real for real arguments, we find $\text{Im}M(c; E + i0) = 0$, $E < 0$.

Let $E = p^2 \geq 0$, $\beta = \sqrt{E} = p \geq 0$. Using the representation $H_\nu^{(1)}(px) = J_\nu(px) + iN_\nu(px)$, where N_ν is the Neumann function, which is real for real arguments, we find

$$\text{Im}M(c; E + i0) = \frac{\pi}{2} c J_\nu^2(\sqrt{E}c) = \frac{\pi}{2} (E/4k_0^2)^\nu [u_1(c; E)]^2, \quad E \geq 0. \quad (59)$$

Using (59), we obtain that $\sigma'(E) = 2^{-1}(E/4k_0^2)^\nu$, $E \geq 0$.

This means that the energy spectrum of the s.a. Calogero Hamiltonian \hat{H}_1 is the semiaxis \mathbb{R}_+ , $\text{spec } \hat{H}_1 = [0, \infty)$, is continuous and simple.

The normalized generalized eigenfunctions

$$u_{1,E}(x) = \sqrt{\rho_1(E)} u_1(x; E) = \frac{1}{\sqrt{2}} x^{1/2} J_\nu(\sqrt{E}x), \quad E \geq 0, \quad (60)$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian \hat{H}_1 .

We note that the inversion formulas in this case coincide with the known formulas for the Fourier–Bessel transformation; see, for example [33, 34].

4.3. Second region: $-1/4 < \alpha < 3/4$ ($0 < \nu < 1$)

For each α in this region, there exists a one-parameter $U(1)$ -family of s.a. Calogero Hamiltonians $\hat{H}_{2,\lambda}$, $|\lambda| \leq \infty$, defined on the domains $D_{H_{2,\lambda}}$ given by (23).

We first note that the function

$$u_2(x; W) = (\beta/2k_0)^\nu x^{1/2} J_{-\nu}(\beta x) \quad (61)$$

is a solution of the homogeneous equation (30), linearly independent of the solution u_1 (48) (of course, with the new value of ν) and is real entire in W . Its asymptotic behavior at the origin, as $x \rightarrow 0$, is given by

$$u_2(x; W) = \frac{k_0^{-1/2}}{\Gamma(1-\nu)} (k_0 x)^{1/2-\nu} [1 + O(x^2)], \quad (62)$$

and, therefore, taking (50) and the relation $\Gamma(1+\nu)\Gamma(1-\nu) = \pi\nu / \sin \pi\nu$ into account, we have $\text{Wr}(u_1, u_2) = -2\pi^{-1} \sin \pi\nu$.

According to the a.b. conditions (22), we have to distinguish the cases of $|\lambda| < \infty$ and $|\lambda| = \infty$. We first consider the case of $|\lambda| < \infty$.

The a.b. conditions (22) with $|\lambda| < \infty$, formulas (50) and (62), on the one hand, and formulas (49) and (52), representation (44), and the relation

$$H_\nu^{(1)}(z) = \frac{1}{i \sin \pi\nu} [J_{-\nu}(z) - e^{-i\pi\nu} J_\nu(z)], \quad (63)$$

on the other hand, define the following choice for the functions u , \tilde{u} and v (see Preliminary) in the case under consideration:

$$\begin{aligned} u_{2,\lambda}(x; W) &= u_1(x, W) + \tilde{\lambda} u_2(x, W) \\ &= (\beta/2k_0)^{-\nu} x^{1/2} J_\nu(\beta x) + \tilde{\lambda} (\beta/2k_0)^\nu x^{1/2} J_{-\nu}(\beta x), \end{aligned} \quad (64)$$

$$\tilde{u}_2(x; W) = u_2(x, W) = (\beta/2k_0)^\nu x^{1/2} J_{-\nu}(\beta x),$$

¹⁸ Other names are the Bessel functions of imaginary arguments.

$$\begin{aligned}
 v_2(x; W) &= \frac{\sin \pi \kappa}{i} e^{+i\pi \kappa} (\beta/2k_0)^{-\kappa} x^{1/2} H_x^{(1)}(\beta x) \\
 &= u_1(x, W) - (e^{-i\pi/2} \beta/2k_0)^{-2\kappa} u_2(x, W) \\
 &= u_{2,\lambda}(x, W) - [\tilde{\lambda} + (e^{-i\pi/2} \beta/2k_0)^{-2\kappa}] \tilde{u}_2(x, W), \quad \tilde{\lambda} = \frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \lambda. \quad (65)
 \end{aligned}$$

We note that $\text{sign } \tilde{\lambda} = \text{sign } \lambda$.

It is easy to see that the function $u_{2,\lambda}(x; W)$ is real entire in W and its asymptotic behavior at the origin is given by

$$u_{2,\lambda}(x; W) = \frac{k_0^{-1/2}}{\Gamma(1+\kappa)} [(k_0 x)^{1/2+\kappa} + \lambda (k_0 x)^{1/2-\kappa}] [1 + O(x^2)], \quad x \rightarrow 0$$

(which agrees with the required a.b. conditions) and that $\tilde{\omega}_2 = -\text{Wr}(u_{2,\lambda}, \tilde{u}_2) = 2\pi^{-1} \sin \pi \kappa$ and

$$\begin{aligned}
 \omega_{2,\lambda}(W) &= -\text{Wr}(u_{2,\lambda}, v_2) = -[\tilde{\lambda} + (e^{-i\pi/2} \beta/2k_0)^{-2\kappa}] \tilde{\omega}_2 \\
 &= -\frac{2 \sin \pi \kappa}{\pi} [\tilde{\lambda} + (e^{-i\pi/2} \beta/2k_0)^{-2\kappa}]. \quad (66)
 \end{aligned}$$

In addition, the last equality in (65) is a copy of the required representation (44) with the evident substitutions $v \rightarrow v_2$, $u \rightarrow u_{2,\lambda}$, $\omega \rightarrow \omega_{2,\lambda}$, $\tilde{\omega} \rightarrow \tilde{\omega}_2$, $\tilde{u} \rightarrow \tilde{u}_2$.

The guiding functional is given by

$$\Phi(\xi; W) = \int_0^\infty u_{2,\lambda}(y; W) \xi(y) dy, \quad \xi \in D_{H_{2,\lambda}} \cap D_r(\mathbb{R}_+).$$

The proof of the simplicity of this guiding functional is completely similar to that in the previous subsection 4.2 for the first region of values of α with the replacements $u_1(x, W) \rightarrow u_{2,\lambda}(x; W)$ and $v_1(x; W) \rightarrow v_2(x; W)$.

It follows that a copy of representation (45) holds true for the spectral density, $\sigma'(E) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \text{Im } \omega_{2,\lambda}^{-1}(E + i\varepsilon)$.

The function $\omega_{2,\lambda}(E) = \omega_{2,\lambda}(E + i0)$ is given by

$$\omega_{2,\lambda}(E) = -\frac{2 \sin \pi \kappa}{\pi} \begin{cases} [\tilde{\lambda} + (-E/4k_0^2)^{-\kappa}], & E < 0, \\ (E/4k_0^2)^{-\kappa} [\cos \pi \kappa + \tilde{\lambda} (E/4k_0^2)^\kappa + i \sin \pi \kappa], & E \geq 0, \end{cases} \quad (67)$$

which shows that the function $\omega_{2,\lambda}(E)$ is real for $E < 0$ and has only one negative simple zero if $\lambda < 0$, while for $E \geq 0$, it is nonzero and complex valued. We can, therefore, use formulas (46) and (47) with $\omega(E) = \omega_{2,\lambda}(E)$ for evaluating the spectral density $\sigma'(E)$, but have to distinguish two regions of the values of the extension parameter λ : $\lambda \geq 0$ and $\lambda < 0$.

(1) Let $\lambda \geq 0$. We then find

$$\sigma'(E) = \frac{\theta(E)}{2\zeta_{2,\lambda}(E)} (E/4k_0^2)^\kappa, \quad \lambda \geq 0, \quad \theta(E) = \begin{cases} 1, & E \geq 0, \\ 0, & E < 0, \end{cases} \quad (68)$$

where

$$\zeta_{2,\lambda}(E) = 1 + 2\tilde{\lambda} (E/4k_0^2)^\kappa \cos \pi \kappa + \tilde{\lambda}^2 (E/4k_0^2)^{2\kappa}. \quad (69)$$

This means that the energy spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda \geq 0$ is the semiaxis \mathbb{R}_+ , $\text{spec } \hat{H}_{2,\lambda} = [0, \infty)$, is continuous and simple, as well as the spectrum of \hat{H}_1 .

The generalized eigenfunctions are $u_{2,\lambda}(x; E)$, $E \geq 0$, given by (64) with the substitution $W = E$ and $\beta = \sqrt{E}$,

$$u_{2,\lambda}(x; E) = (E/4k_0^2)^{-x/2} x^{1/2} J_x(\sqrt{E}x) + \tilde{\lambda}(E/4k_0^2)^{x/2} x^{1/2} J_{-x}(\sqrt{E}x), \quad E \geq 0. \quad (70)$$

The normalized generalized eigenfunctions

$$u_{2,\lambda,E}(x) = \frac{1}{\sqrt{2}} \frac{x^{1/2} J_x(\sqrt{E}x) + \gamma(\lambda, E)x^{1/2} J_{-x}(\sqrt{E}x)}{\sqrt{1 + 2\gamma(\lambda, E) \cos \pi x + \gamma^2(\lambda, E)}}, \quad (71)$$

$$\gamma(\lambda, E) = \lambda \frac{\Gamma(1-x)}{\Gamma(1+x)} (E/4k_0^2)^x, \quad \lambda \geq 0, \quad E \geq 0,$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda \geq 0$.

For $\lambda = 0$, the inversion formulas coincide with the formulas for the Fourier–Bessel transformation.

- (2) Let $\lambda < 0$. The only difference from the case of $\lambda > 0$ is that the function $\omega_{2,\lambda}(E)$, given by the same equation (67), now has a unique simple zero $E_{2,\lambda}$, $\omega_{2,\lambda}(E_{2,\lambda}) = 0$, in the negative energy region,

$$E_{2,\lambda} = -4k_0^2 \left| \lambda \frac{\Gamma(1-x)}{\Gamma(1+x)} \right|^{-1/x}, \quad (72)$$

and¹⁹

$$\frac{d\omega_{2,\lambda}(E_{2,\lambda})}{dE} = -\frac{2x \sin \pi x}{\pi} \frac{\tilde{\lambda}}{E_{2,\lambda}}.$$

Therefore, (68) changes to

$$\sigma'(E) = \frac{\pi}{2x \sin \pi x} \frac{E_{2,\lambda}}{\tilde{\lambda}} \delta(E - E_{2,\lambda}) + \frac{\theta(E)}{2\zeta_{2,\lambda}(E)} (E/4k_0^2)^x, \quad \lambda < 0,$$

and $\zeta_{2,\lambda}(E)$ is given by the same relation (69), of course, with $\lambda < 0$.

This means that the energy spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda < 0$ is the union of a discrete spectrum, the negative energy level $E_{2,\lambda}$ (72) corresponding to a bound state, and a continuous spectrum, the semiaxis \mathbb{R}_+ ,

$$\text{spec } \hat{H}_{2,\lambda} = \left\{ -4k_0^2 \left| \lambda \frac{\Gamma(1-x)}{\Gamma(1+x)} \right|^{-1/x} \right\} \cup [0, \infty), \quad \lambda < 0.$$

For the bound-state eigenfunction, we take²⁰ $u_{2,\lambda}(x; E_{2,\lambda}) = v_2(x; E_{2,\lambda})$; the last equality follows from (65) with $W = E_{2,\lambda}$: the second term on the rhs of the last equality in (65) vanishes because it is proportional to $\omega_{2,\lambda}(E_{2,\lambda}) = 0$. Using the second relation in (58) and (65), we find

$$u(x, E_{2,\lambda}) = -\frac{2 \sin \pi x}{\pi} |\tilde{\lambda}|^{1/2} x^{1/2} K_x(\sqrt{|E_{2,\lambda}|}x).$$

The generalized eigenfunctions of the continuous spectrum $u_{2,\lambda}(x; E)$ are given by (70) with $\lambda < 0$.

¹⁹ Both $\tilde{\lambda}$ and $E_{2,\lambda}$ are negative, and therefore $d\omega_{2,\lambda}(E_{2,\lambda})/dE$ is also negative.

²⁰ We change the sign for the sake of convenience.

The normalized bound-state eigenfunction

$$\begin{aligned}
 u_{E_{2,\lambda}}(x) &= - \left(\frac{\pi}{2\kappa \sin \pi \kappa} \frac{E_{2,\lambda}}{\tilde{\lambda}} \right)^{1/2} u(x, E_{2,\lambda}) \\
 &= \sqrt{\frac{2 \sin \pi \kappa}{\pi \kappa}} |E_{2,\lambda}|^{1/2} x^{1/2} K_\kappa(\sqrt{|E_{2,\lambda}|}x), \\
 E_{2,\lambda} &= -4k_0^2 \left| \lambda \frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \right|^{-1/\kappa}, \quad \lambda < 0,
 \end{aligned}
 \tag{73}$$

and the normalized generalized eigenfunctions of the continuous spectrum

$$\begin{aligned}
 u_{2,\lambda,E}(x) &= \frac{1}{\sqrt{2}} \frac{x^{1/2} J_\kappa(\sqrt{E}x) + \gamma(\lambda, E) x^{1/2} J_{-\kappa}(\sqrt{E}x)}{\sqrt{1 + 2\gamma(\lambda, E) \cos \pi \kappa + \gamma^2(\lambda, E)}}, \\
 \gamma(\lambda, E) &= \lambda \frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} (E/4k_0^2)^\kappa, \quad \lambda < 0, \quad E \geq 0,
 \end{aligned}
 \tag{74}$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda < 0$.

Apart from the sign of λ , the inversion formulas and the Parseval equality for $\hat{H}_{2,\lambda}$ with $\lambda < 0$ differ from those for $\hat{H}_{2,\lambda}$ with $\lambda \geq 0$ by an additional term stemming from a supplementary bound state.

The inversion formulas with a nonzero λ are known and can be found, for example, in [35], where they are obtained by a different method.

We now turn to the remaining case of $|\lambda| = \infty$, i.e. to the s.a. Calogero Hamiltonian $\hat{H}_{2,\infty}$, specified by a.b. conditions (22) with $|\lambda| = \infty$. Armed with the experience of the preceding consideration, we restrict ourselves to presenting the main items with short comments.

The a.b. conditions (22) with $|\lambda| = \infty$, formulas (61) and (62), on the one hand, and formulas (49) and (52), representation (44), and relation (63), on the other hand, define the following choice for the functions u, \tilde{u} , and v in this case:

$$\begin{aligned}
 u_{2,\infty}(x; W) &= u_2(x; W) = (\beta/2k_0)^\kappa x^{1/2} J_{-\kappa}(\beta x), \\
 \tilde{u}_{2,\infty}(x; W) &= u_1(x; W) = (\beta/2k_0)^{-\kappa} x^{1/2} J_\kappa(\beta x), \\
 v_{2,\infty}(x; W) &= i \sin \pi \kappa (\beta/2k_0)^\kappa x^{1/2} H_\kappa^{(1)}(\beta x) \\
 &= u_{2,\infty}(x; W) - e^{-i\pi \kappa} (\beta/2k_0)^{2\kappa} \tilde{u}_{2,\infty}(x; W),
 \end{aligned}$$

with $\tilde{\omega}_{2,\infty} = -\text{Wr}(u_{2,\infty}, \tilde{u}_{2,\infty}) = -2\pi^{-1} \sin \pi \kappa$, and

$$\omega_{2,\infty}(W) = -\text{Wr}(u_{2,\infty}, v_{2,\infty}) = -e^{-i\pi \kappa} (\beta/2k_0)^{2\kappa} \tilde{\omega}_{2,\infty} = 2 \frac{\sin \pi \kappa}{\pi} e^{-i\pi \kappa} (\beta/2k_0)^{2\kappa}.$$

The last equality in (75) is a copy of the required representation (44) with the evident substitutions $v \rightarrow v_{2,\infty}$, $u \rightarrow u_{2,\infty}$, $\tilde{u} \rightarrow \tilde{u}_{2,\infty}$, $\omega \rightarrow \omega_{2,\infty}$, $\tilde{\omega} \rightarrow \tilde{\omega}_{2,\infty}$, $u \rightarrow u_{2,\infty}$.

The guiding functional is given by

$$\Phi(\xi; W) = \int_0^\infty u_{2,\infty}(x; W) \xi(x) dx, \quad \xi \in D_{H_{2,\infty}} \cap D_r(\mathbb{R}_+).$$

It is simple, which is proved as in the previous subsection 4.2.

It follows that the spectral density is given by $\sigma'(E) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \text{Im} \omega_{2,\lambda}^{-1}(E + i\varepsilon)$. The function $\omega_{2,\infty}(E) = \omega_{2,\infty}(E + i0)$ is given by

$$\omega_{2,\infty}(E) = 2 \frac{\sin \pi \kappa}{\pi} \begin{cases} (-E/4k_0^2)^\kappa, & E < 0, \\ e^{-i\pi \kappa} (E/4k_0^2)^\kappa, & E \geq 0. \end{cases}$$

It is a nonzero real function for $E < 0$ and a nonzero complex-valued function for $E > 0$, which allows applying formulas (46) and (47) with $\omega(E) = \omega_{2,\infty}(E)$ for evaluating the spectral density $\sigma'(E)$, and we find that $\sigma'(E) = 2^{-1}\theta(E)(E/4k_0^2)^{-\alpha}$.

This means that the energy spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{2,\infty}$ is the semiaxis \mathbb{R}_+ , $\text{spec } \hat{H}_{2,\infty} = [0, \infty)$, is continuous and simple.

The generalized eigenfunctions are $u_{2,\infty}(x; E)$, $E \geq 0$.

The normalized eigenfunctions

$$u_{2,\infty;E}(x) = \frac{1}{\sqrt{2}}x^{1/2}J_{-\alpha}(\sqrt{E}x), \quad E \geq 0,$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{2,\infty}$.

We note that, with the evident changes $u_{2,\lambda} \rightarrow \lambda^{-1}u_{2,\lambda}$ and $v_2 \rightarrow \lambda^{-1}v_2$, all the results can be obtained from the previous results for $|\lambda| < \infty$ by the formal passage to the limit $|\lambda| \rightarrow \infty$.

The respective inversion formulas coincide with the formulas for the Fourier–Bessel transformation that is known for the indices of the Bessel functions larger than -1 and do not hold for the indices equal to or less than -1 .

Concluding this subsection, we make some remarks for physicists.

It is interesting to note that for $\lambda \geq 0$ and $|\lambda| = \infty$ there is no bound states even if the coupling constant α is negative, $-1/4 < \alpha < 0$, so that the Calogero potential is attractive, while for any finite $\lambda < 0$, a single bound state exists even if α is nonnegative, $0 \leq \alpha < 3/4$, so that the Calogero potential is zero or repulsive, and as λ changes in the interval $(-\infty, 0)$, the bound-state energy $E_{2,\lambda}$ ranges between 0 and $-\infty$. If the Calogero Hamiltonian is treated as the s -wave radial Hamiltonian for the three-dimensional motion, we can suggest that these phenomena may be interpreted as a manifestation of δ -like potentials at the origin. In addition, we emphasize that a s.a. Hamiltonian with $\alpha = 0$, treated as a QM Hamiltonian for a free motion of a nonrelativistic particle on a semiaxis, is not uniquely defined: there exists a $U(1)$ family of such Hamiltonians specified by different a.b. conditions at the origin. In particular, a negative energy level $E_{2,\lambda}$, $\lambda < 0$, can be treated as a Tamm level; see [36].

4.4. Third region: $\alpha = -1/4$ ($\alpha = 0$)

For $\alpha = -1/4$, there exists a one-parameter $U(1)$ family of s.a. Calogero Hamiltonians $\hat{H}_{3,\lambda}$, $|\lambda| \leq \infty$, defined on the domains $D_{H_{3,\lambda}}$ given by (26).

The spectral analysis for the Hamiltonian $\hat{H}_{3,\lambda}$ is similar to that for the Hamiltonian $\hat{H}_{2,\lambda}$ based on representations (44) and (45) and presented in detail in the previous subsection. We will therefore dwell only on specific features of the case.

A specific feature of the case under consideration is that the functions $u_1(x; W)$ (48) and $u_2(x; W)$ (61) with $\alpha = 0$ coincide, and therefore, we have to find an alternative to the function $u_2(x; W)$.

We note that the function $x^{1/2}N_0(\beta x)$ is a solution of equation (30) with $\alpha = -1/4$ linearly independent of the solution $u_1(x; W) = x^{1/2}J_0(\beta x)$ and recall that the relation

$$\frac{\pi}{2}N_0(z) = (\ln z + \mathbf{C})J_0(z) - R_0(z) \tag{76}$$

holds true, where \mathbf{C} is the Euler constant and

$$R_0(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2k} \sum_{m=1}^k \frac{1}{m}$$

is a function real entire in z^2 . It follows that the function

$$u_3(x; W) = x^{1/2} \left[\frac{\pi}{2} N_0(\beta x) - (\ln \beta/2k_0 + \mathbf{C}) J_0(\beta x) \right] \\ = x^{1/2} [J_0(\beta x) \ln(k_0 x) - R_0(\beta x)],$$

where $\ln(\beta/2k_0) = \ln(\sqrt{|W|}/2k_0) + i\varphi/2$, is a solution of equation (30) with $\alpha = -1/4$, which is linearly independent of the solution $u_1(x; W)$ and is real entire in W , as well as $u_1(x; W)$. The asymptotic behavior of this function at the origin is given by

$$u_3(x; W) = x^{1/2} \ln(k_0 x) + O(x^{3/2} \ln x), \quad x \rightarrow 0,$$

whence it follows in particular, together with (50), $\kappa = 0$, that $\text{Wr}(u_3, u_1) = -1$.

The a.b. conditions (25) send us to distinguish the cases of $|\lambda| < \infty$ and $|\lambda| = \infty$.

We first consider the case of $|\lambda| < \infty$.

The a.b. conditions (25) with $|\lambda| < \infty$, the known properties of the functions u_1, u_3 , and v_1 (49) with $\kappa = 0$, representation (44) and the relations $H_0^{(1)}(z) = J_0(z) + iN_0(z)$ and (76) define the following choice for the functions u, \tilde{u} and v in this case:

$$u_{3,\lambda}(x; W) = \lambda u_1(x, W) + u_3(x, W) \\ = \lambda x^{1/2} J_0(\beta x) + x^{1/2} \left[\frac{\pi}{2} N_0(\beta x) - (\ln \beta/2k_0 + \mathbf{C}) J_0(\beta x) \right] \\ = x^{1/2} \left[\tilde{\lambda}(W) J_0(\beta x) + \frac{\pi}{2} N_0(\beta x) \right], \tag{77}$$

$$\tilde{u}_3(x; W) = u_1(x, W), \\ v_3(x; W) = -\frac{i\pi}{2} x^{1/2} H_0^{(1)}(\beta x) = x^{1/2} \left[\frac{\pi}{2} N_0(\beta x) - \frac{i\pi}{2} J_0(\beta x) \right] \\ = u_{3,\lambda}(x, W) - \left(\tilde{\lambda}(W) + i\frac{\pi}{2} \right) \tilde{u}_3(x, W), \tag{78}$$

where $\tilde{\lambda}(W) = \lambda - \mathbf{C} - \ln \beta/2k_0$, with $\tilde{\omega}_3 = -\text{Wr}(u_{3,\lambda}, \tilde{u}_3) = -\text{Wr}(u_3, u_1) = 1$ and

$$\omega_{3,\lambda}(W) = -\text{Wr}(u_{3,\lambda}, v_3) = -\left(\tilde{\lambda}(W) + i\frac{\pi}{2} \right) \tilde{\omega}_3 \\ = -\left(\tilde{\lambda}(W) + i\frac{\pi}{2} \right) = \ln \beta/2k_0 + \mathbf{C} - \lambda - i\frac{\pi}{2}.$$

The last equality in (78) is the required copy of representation (44) for v_3 .

The guiding functional is given by

$$\Phi(\xi; W) = \int_0^\infty u_{3,\lambda}(y; W) \xi(y) dy, \quad \xi \in D_{H_{3,\lambda}} \cap D_r(\mathbb{R}_+).$$

It is simple, which is proved similar to subsection 4.2.

It follows that the spectral density is given by $\sigma'(E) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \text{Im} \omega_{3,\lambda}^{-1}(E + i\varepsilon)$. The function $\omega_{3,\lambda}(E) = \omega_{3,\lambda}(E + i0)$ given by

$$\omega_{3,\lambda}(E) = \begin{cases} \frac{1}{2} \ln(-E/4k_0^2) + \mathbf{C} - \lambda, & E < 0, \\ \frac{1}{2} \ln(E/4k_0^2) + \mathbf{C} - \lambda - i\frac{\pi}{2}, & E > 0, \end{cases}$$

is real on the negative semiaxis and has a single simple zero $E_{3,\lambda} = -4k_0^2 e^{2(\lambda - \mathbf{C})}$, $\omega_{3,\lambda}(E_{3,\lambda}) = 0$, with $\omega'_{3,\lambda}(E_{3,\lambda}) = (2E_{3,\lambda})^{-1}$, while on the semiaxis \mathbb{R}_+ , it is a nonzero complex-valued function. This allows using formulas (46) and (47) for evaluating the spectral density to yield

$$\sigma'(E) = 2(-E_{3,\lambda})\delta(E - E_{3,\lambda}) + \frac{\theta(E)}{2\zeta_{3,\lambda}(E)},$$

where

$$\zeta_{3,\lambda}(E) = \tilde{\lambda}^2(E) + \frac{\pi^2}{4} = \left(\frac{1}{2} \ln(E/4k_0^2) + C - \lambda\right)^2 + \frac{\pi^2}{4}.$$

This means that the simple energy spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{3,\lambda}$ is the union of a discrete spectrum, the negative energy level $E_{3,\lambda}$ corresponding to a bound state, and a continuous spectrum, the semiaxis \mathbb{R}_+ , $\text{spec } \hat{H}_{2,\lambda} = \{-4k_0^2 e^{2(\lambda-C)}\} \cup [0, \infty)$.

We note that in contrast to the second region of values of α , $-1/4 < \alpha < 3/4$, a bound state in the case of $\alpha = -1/4$ exists for any finite λ , and, as λ changes in the interval $(-\infty, \infty)$, the bound-state energy $E_{3,\lambda}$ ranges between 0 and $-\infty$. In some sense, it is mysterious if we treat the Calogero Hamiltonian with $\alpha = -1/4$ as an s -wave radial Hamiltonian for a free particle in two dimensions and the extension parameter λ as a manifestation of a δ -like potential that can have any sign.

The normalized bound-state eigenfunction

$$\begin{aligned} u_{E_{3,\lambda}}(x) &= -\sqrt{2}|E_{3,\lambda}|^{1/2}u_{3,\lambda}(x; E_{3,\lambda}) = -\sqrt{2}|E_{3,\lambda}|^{1/2}v_3(x; E_{3,\lambda}) \\ &= \sqrt{2}|E_{3,\lambda}|^{1/2}x^{1/2}K_0(\sqrt{|E_{3,\lambda}|}x), \quad E_{3,\lambda} = -4k_0^2 e^{2(\lambda-C)}, \end{aligned}$$

and the normalized generalized eigenfunctions of the continuous spectrum

$$\begin{aligned} u_{3,\lambda,E}(x) &= \frac{1}{\sqrt{2\zeta_{3,\lambda}(E)}}x^{1/2}\left[\tilde{\lambda}(E)J_0(\sqrt{E}x) + \frac{\pi}{2}N_0(\sqrt{E}x)\right], \\ \zeta_{3,\lambda}(E) &= \tilde{\lambda}^2(E) + \frac{\pi^2}{4}, \quad \tilde{\lambda}(E) = \lambda - C - \frac{1}{2} \ln(E/4k_0^2), \quad E \geq 0, \end{aligned}$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{2,\infty}$.

As to the remaining case of $|\lambda| = \infty$, we only outline the main points.

The functions $u_{3,\lambda}$ (77) and v_3 (78) in the previous case of $|\lambda| < \infty$ are evidently replaced by the respective functions

$$u_{3,\infty}(x; W) = u_1(x, W) = x^{1/2}J_0(\beta x)$$

and

$$v_{3,\infty}(x, W) = \frac{x^{1/2}H_0^{(1)}(\beta x)}{1 + \frac{2i}{\pi}(\ln \beta/2k_0 + C)} = u_1(x, W) + \frac{u_3(x; W)}{(\ln \beta/2k_0 + C) - \frac{i\pi}{2}}.$$

The guiding functional given by

$$\Phi(\xi; W) = \int_0^\infty u_1(y; W)\xi(y) dy, \quad \xi \in D_{H_{3,\infty}} \cap D_r(\mathbb{R}_+)$$

is simple; the proof is similar to that in subsection 4.2.

The spectral density is $\sigma'(E) = 2^{-1}\theta(E)$. This means that the spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{3,\infty}$ is the semiaxis \mathbb{R}_+ , $\text{spec } \hat{H}_{3,\infty} = [0, \infty)$; it is continuous and simple.

The normalized generalized eigenfunctions

$$u_{3,\infty;E}(x) = \frac{1}{\sqrt{2}}x^{1/2}J_0(\sqrt{E}x), \quad E \geq 0,$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{3,\infty}$.

With the evident changes $u_{3,\lambda} \rightarrow \lambda^{-1}u_{3,\lambda}$ and $v_3 \rightarrow \lambda^{-1}v_3$, all the results can be obtained from the previous results for $|\lambda| < \infty$ by the formal passage to the limit $|\lambda| \rightarrow \infty$.

The respective inversion formulas coincide with the standard formulas for the Fourier-Bessel transformation.

4.5. Fourth region: $\alpha < -1/4$ ($\kappa = i\sigma$, $\sigma > 0$)

For each α in this region, there exists a one-parameter $U(1)$ family of s.a. Calogero Hamiltonians $\hat{H}_{4,\theta}$, $0 \leq \theta \leq \pi$, $\theta = 0 \sim \theta = \pi$, defined on the domains $D_{H_{4,\theta}}$ given by (28).

The spectral analysis for the Hamiltonian $\hat{H}_{4,\theta}$ is completely similar to that for the Hamiltonians $\hat{H}_{2,\lambda}$ and $\hat{H}_{3,\lambda}$ in the previous two subsections based on representations (44) and (45), and we therefore concentrate only on distinctive features of the case.

The first distinctive feature is that the two linearly independent solutions u_1 (48) and u_2 (61) of equation (30) with $\alpha < -1/4$ ($\kappa = i\sigma$) are no longer real entire: they are entire in W , but complex conjugate on the real axis, $u_2(x; E) = \overline{u_1(x; E)}$. Relevant linearly independent real-entire solutions are

$$\begin{aligned} u_{+,\vartheta}(x, W) &= e^{i\vartheta} u_1(x, W) + e^{-i\vartheta} u_2(x, W), \\ u_{-,\vartheta}(x, W) &= i(e^{-i\vartheta} u_2(x, W) - e^{i\vartheta} u_1(x, W)), \end{aligned}$$

with $\text{Wr}(u_{+,\vartheta}, u_{-,\vartheta}) = 2i\text{Wr}(u_1, u_2) = 4\pi^{-1} \sinh \pi\kappa$, for any fixed $\vartheta = \bar{\vartheta}$ considered mod 2π . Accordingly,

$$\begin{aligned} u_1(x, W) &= \frac{1}{2} e^{-i\vartheta} [u_{+,\vartheta}(x, W) + i u_{-,\vartheta}(x, W)], \\ u_2(x, W) &= \frac{1}{2} e^{i\vartheta} [u_{+,\vartheta}(x, W) - i u_{-,\vartheta}(x, W)]. \end{aligned}$$

For the functions u , \tilde{u} , and v in this case, we take

$$\begin{aligned} u_{4,\theta}(x, W) &= u_{+,\tilde{\theta}}(x, W) \\ &= e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma} x^{1/2} J_{i\sigma}(\beta x) + e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} x^{1/2} J_{-i\sigma}(\beta x), \\ \tilde{u}_{4,\theta}(x, W) &= u_{-,\tilde{\theta}}(x, W) \\ &= i[e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} x^{1/2} J_{-i\sigma}(\beta x) - e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma} x^{1/2} J_{i\sigma}(\beta x)], \\ v_{4,\theta}(x, W) &= \frac{2 \sinh \pi\sigma}{e^{\pi\sigma} e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} - e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma}} x^{1/2} H_{i\sigma}(\beta x) \\ &= u_{\theta}(x, W) + i \frac{e^{\pi\sigma} e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} + e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma}}{e^{\pi\sigma} e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} - e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma}} \tilde{u}_{\theta}(x, W), \end{aligned} \tag{79}$$

where

$$\tilde{\theta} = \theta + \theta_{\sigma}, \quad \theta_{\sigma} = \frac{1}{2i} \ln \frac{\Gamma(1 + i\sigma)}{\Gamma(1 - i\sigma)},$$

with $\tilde{\omega}_4 = -\text{Wr}(u_{4,\theta}, \tilde{u}_{4,\theta}) = -4\pi^{-1} \sinh \pi\sigma$ and

$$\omega_{4,\theta}(W) = -\text{Wr}(u_{4,\theta}, v_{4,\theta}) = -i4\pi^{-1} \sinh \pi\sigma \frac{e^{\pi\sigma} e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} + e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma}}{e^{\pi\sigma} e^{-i\tilde{\theta}} (\beta/2k_0)^{i\sigma} - e^{i\tilde{\theta}} (\beta/2k_0)^{-i\sigma}}.$$

It is easy to see or check that these functions possess all the required properties; in particular, the last equality in (79), being a copy of representation (44), follows from relation (63) with $\kappa = i\sigma$.

The guiding functional given by

$$\Phi(\xi; W) = \int_0^{\infty} u_{4,\theta}(y; W) \xi(y) dy, \quad \xi \in D_{H_{4,\theta}} \cap D_r(\mathbb{R}_+)$$

is simple, which is proved similar to subsection 4.2.

It follows that the spectral density is given by $\sigma'(E) = \pi^{-1} \lim_{\varepsilon \rightarrow +0} \text{Im } \omega_{4,\theta}^{-1}(E + i\varepsilon)$. The function $\omega_{4,\theta}(E) = \omega_{4,\theta}(E + i0)$ is given by

$$\omega_{4,\theta}(E) = -\frac{4 \sinh \pi \sigma}{\pi} \begin{cases} \cot \frac{1}{2} \Phi_\theta(-E), & E < 0, \\ i \frac{e^{\pi\sigma} + e^{-i\Phi_\theta(E)}}{e^{\pi\sigma} - e^{-i\Phi_\theta(E)}}, & E > 0, \end{cases}$$

where $\Phi_\theta(E) = \sigma \ln(E/4k_0^2) - 2\tilde{\theta}$ is real for $E > 0$. On the negative semiaxis, this function is real and has an infinite sequence $\{E_{\theta,n}, n \in \mathbb{Z}\}$ of simple zeroes, $\omega_{4,\theta}(E_{\theta,n}) = 0$,

$$E_{\theta,n} = -4k_0^2 \exp\left(2 \frac{\pi/2 + \tilde{\theta} + \pi n}{\sigma}\right), \quad n \in \mathbb{Z}, \tag{80}$$

with

$$\omega'_{4,\theta}(E_{\theta,n}) = -\frac{2\sigma \sinh \pi \sigma}{\pi |E_{\theta,n}|},$$

while on the positive semiaxis it is nonzero and complex-valued. A simple calculation by formulas (46) and (47) then yields

$$\sigma'(E) = \sum_{n=-\infty}^{\infty} \frac{\pi |E_{\theta,n}|}{2\sigma \sinh(\pi\sigma)} \delta(E - E_{\theta,n}) + \frac{\theta(E)}{4 [\cosh \pi\sigma + \cos \Phi_\theta(E)]}.$$

This means that the simple energy spectrum of the s.a. Calogero Hamiltonian $\hat{H}_{4,\theta}$ is the union of a discrete spectrum, the infinite sequence $\{E_{\theta,n}, n \in \mathbb{Z}\}$ of negative energy levels $E_{\theta,n}$ (80) corresponding to bound states²¹, and a continuous spectrum, the semiaxis \mathbb{R}_+ ,

$$\text{spec } \hat{H}_{4,\theta} = \left\{ -4k_0^2 \exp\left(2 \frac{\pi/2 + \tilde{\theta} + \pi n}{\sigma}\right), n \in \mathbb{Z} \right\} \cup [0, \infty).$$

The negative energy levels are accumulated exponentially to zero as $n \rightarrow -\infty$ and go exponentially to $-\infty$ as $n \rightarrow \infty$, so that the energy spectrum for all s.a. Calogero Hamiltonians with $\alpha < -1/4$ is not bounded from below. The radius of the bound states goes to zero as $n \rightarrow \infty$, which manifests the phenomenon of a ‘fall to the center’.

Accordingly, the normalized bound-state eigenfunctions²²

$$\begin{aligned} u_{E_{\theta,n}}(x) &= (-1)^{n+1} \left(\frac{\pi |E_{\theta,n}|}{2\sigma \sinh \pi \sigma} \right)^{1/2} u_{4,\theta}(x; E_{\theta,n}) \\ &= \left(\frac{2 \sinh \pi \sigma |E_{\theta,n}|}{\pi \sigma} \right)^{1/2} x^{1/2} K_{i\sigma}(|E_{\theta,n}|^{1/2} x), \end{aligned} \tag{81}$$

$$E_{\theta,n} = -4k_0^2 \exp \frac{2}{\sigma} (\pi/2 + \tilde{\theta} + \pi n), \quad n \in \mathbb{Z},$$

and the normalized generalized eigenfunctions of the continuous spectrum

$$\begin{aligned} u_{4,\theta;E}(x) &= \sqrt{\rho_{4,\theta}(E)} u_{4,\theta}(x; E) = \frac{1}{2\sqrt{\cosh \pi \sigma + \cos \Phi_\theta(E)}} \\ &\times [e^{i\tilde{\theta}} (E/4k_0^2)^{-i\sigma/2} x^{1/2} J_{i\sigma}(\sqrt{E}x) + e^{-i\tilde{\theta}} (E/4k_0^2)^{i\sigma/2} x^{1/2} J_{-i\sigma}(\sqrt{E}x)], \end{aligned} \tag{82}$$

$$\Phi_\theta(E) = \sigma \ln E/4k_0^2 - 2\tilde{\theta}, \quad E \geq 0,$$

form a complete orthonormalized system of eigenfunctions for the s.a. Calogero Hamiltonian $\hat{H}_{4,\theta}$.

We have not succeeded in finding a respective inversion formulas in mathematical handbooks. These are an extension of the Fourier–Bessel transformation to imaginary indices of the Bessel functions.

²¹ It is this discrete spectrum that was first presented in [17].

²² The sign factors are introduced for the sake of convenience. We also use the second relation in (58) with $\kappa = i\sigma$.

5. Fate of scale symmetry

The scale parameter k_0 , introduced for dimensional reasons, appears to be significant in s.a. extensions for $\alpha < 3/4$: its change $k_0 \rightarrow lk_0$ generally changes the extension parameter, which indicates the breaking of scale symmetry.

From the mathematical standpoint, it is convenient to parameterize s.a. extensions by a dimensionless parameter, λ or θ . However, from the physical standpoint, it seems more appropriate to convert the two parameters, the fixed dimensional parameter k_0 of spatial dimension $d_{k_0} = -1$ and the varying dimensionless parameters λ and θ of s.a. extensions, to one-dimensional parameter μ of spatial dimension $d_\mu = -1$ uniquely parameterizing the extensions, and the parameter k_0 no longer enters the description. This makes evident the spontaneous breaking of the scale symmetry.

As is easily seen from (22), in the case of $-1/4 < \alpha < 3/4$ and for $\lambda > 0$, this parameter is $\mu = k_0 \lambda^{-\frac{1}{2\alpha}}$, $0 < \mu < \infty$. The s.a. Calogero Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda > 0$ is now naturally labeled by the subscript μ and an extra subscript + indicating the sign of λ , $\hat{H}_{2,\mu,+} = \hat{H}_{2,\lambda}$, $\lambda > 0$, and is specified by the a.b. conditions

$$\begin{aligned} \psi_{2,\mu,+}(x) &= cx^{1/2}[(\mu x)^\kappa + (\mu x)^{-\kappa}] + O(x^{3/2}), \\ \psi'_{2,\mu,+}(x) &= cx^{-1/2}[(1/2 + \kappa)(\mu x)^\kappa + (1/2 - \kappa)(\mu x)^{-\kappa}] + O(x^{1/2}), \quad x \rightarrow 0. \end{aligned} \tag{83}$$

The complete orthonormalized system (71) of eigenfunctions for the Hamiltonian $\hat{H}_{2,\mu,+}$ is presented in terms of the scale parameter μ as follows:

$$\begin{aligned} u_{2,\mu,+;E}(x) &= \frac{1}{\sqrt{2}} \frac{x^{1/2} J_\kappa(\sqrt{E}x) + \gamma_+(\mu, E)x^{1/2} J_{-\kappa}(\sqrt{E}x)}{\sqrt{1 + 2\gamma_+(\mu, E) \cos \pi\kappa + \gamma_+^2(\mu, E)}}, \\ \gamma_+(\mu, E) &= \frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} (E/4\mu^2)^\kappa, \quad E \geq 0; \end{aligned} \tag{84}$$

the auxiliary scale parameter k_0 then disappears.

For $\lambda < 0$, the dimensional parameter is $\mu = k_0 |\lambda|^{-\frac{1}{2\alpha}}$, $0 < \mu < \infty$. The Hamiltonian $\hat{H}_{2,\lambda}$ with $\lambda < 0$ is now denoted by $\hat{H}_{2,\mu,-}$: $\hat{H}_{2,\mu,-} = \hat{H}_{2,\lambda}$, $\lambda < 0$, and is specified by the a. b. conditions

$$\begin{aligned} \psi_{2,\mu,-}(x) &= cx^{1/2}[(\mu x)^\kappa - (\mu x)^{-\kappa}] + O(x^{3/2}), \\ \psi'_{2,\mu,-}(x) &= cx^{-1/2}[(1/2 + \kappa)(\mu x)^\kappa - (1/2 - \kappa)(\mu x)^{-\kappa}] + O(x^{1/2}), \quad x \rightarrow 0. \end{aligned} \tag{85}$$

The single negative energy level representing its discrete spectrum is now given by

$$E_{2,\mu,-} = -4\mu^2 \left(\frac{\Gamma(1 + \kappa)}{\Gamma(1 - \kappa)} \right)^{1/\kappa}. \tag{86}$$

The complete orthonormalized system (73) and (74) of eigenfunctions for the Hamiltonian $\hat{H}_{2,\mu,-}$ is written in terms of the scale parameter μ as

$$\begin{aligned}
 u_{E_{2,\mu,-}}(x) &= \sqrt{\frac{2 \sin \pi \kappa}{\pi \kappa}} (-E_{2,\mu,-})^{1/2} x^{1/2} K_{\kappa}(\sqrt{-E_{2,\mu,-}}x), \\
 u_{2,\mu,+;E}(x) &= \frac{1}{\sqrt{2}} \frac{x^{1/2} J_{\kappa}(\sqrt{E}x) + \gamma_{-}(\mu, E)x^{1/2} J_{-\kappa}(\sqrt{E}x)}{\sqrt{1 + 2\gamma_{-}(\mu, E) \cos \pi \kappa + \gamma_{-}^2(\mu, E)}}, \\
 \gamma_{-}(\mu, E) &= -\frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} (E/4\mu^2)^{\kappa}, \quad E \geq 0,
 \end{aligned}
 \tag{87}$$

We note that the s.a. Calogero Hamiltonian $\hat{H}_{2,\mu,-}$ is uniquely determined by a position of the negative energy level.

The exceptional values $\lambda = 0$ and $|\lambda| = \infty$ of the extension parameter are naturally included in this scheme as the respective exceptional values $\mu = \infty$ and $\mu = 0$ of the scale parameter, and in terms of μ the corresponding Hamiltonians are respectively denoted by $\hat{H}_{2,\infty}$, $\hat{H}_{2,\mu=\infty} = \hat{H}_{2,\lambda=0}$, and $\hat{H}_{2,0}$, $\hat{H}_{2,\mu=0} = \hat{H}_{2,\lambda=\infty}$.

As is seen from (25), in the case of $\alpha = -1/4$ and for $|\lambda| < \infty$, the dimensional parameter is $\mu = k_0 e^{\lambda}$, $0 < \mu < \infty$. In terms of μ , the respective s.a. Calogero Hamiltonian $\hat{H}_{3,\mu}$, $\hat{H}_{3,\mu} = \hat{H}_{\lambda}$, is specified by a.b. conditions

$$\begin{aligned}
 \psi_{3,\mu}(x) &= cx^{1/2} \ln(\mu x) + O(x^{3/2}), \\
 \psi'_{3,\mu}(x) &= cx^{-1/2} \left(\frac{1}{2} \ln(\mu x) + 1\right) + O(x^{1/2}), \quad x \rightarrow 0.
 \end{aligned}
 \tag{88}$$

The single negative energy level representing its discrete spectrum is given by $E_{3,\mu} = -4\mu^2 \exp(-2C)$, where C is the Euler constant; a position of this level uniquely determines the Hamiltonian $\hat{H}_{3,\mu}$.

The exceptional values $\lambda = -\infty$ and $\lambda = \infty$ of the extension parameter λ , which are equivalent, $-\infty \sim \infty$, are naturally included as the respective exceptional values $\mu = \infty$ and $\mu = 0$ of the scale parameter μ , which are equivalent, $\infty \sim 0$. In terms of μ , we let \hat{H}_3 denote the corresponding Hamiltonian, $\hat{H}_3 = \hat{H}_{3,|\lambda|=\infty}$.

As is seen from (27), in the case of $\alpha < -1/4$, the dimensional parameter is

$$\mu = k_0 e^{\frac{\theta}{\sigma}}, \quad \mu_0 \leq \mu \leq \mu_0 e^{\pi/\sigma}, \quad \mu_0 \sim \mu_0 e^{\pi/\sigma}
 \tag{89}$$

with some fixed $\mu_0 > 0$. In terms of μ , the respective s.a. Calogero Hamiltonian $\hat{H}_{4,\mu}$, $\hat{H}_{4,\mu} = \hat{H}_{4,\theta}$, is specified by a.b. conditions

$$\begin{aligned}
 \psi_{4,\mu}(x) &= cx^{1/2} [(\mu x)^{i\sigma} + (\mu x)^{-i\sigma}] + O(x^{3/2}), \\
 \psi'_{4,\mu}(x) &= cx^{-1/2} [(1/2 + i\sigma)(\mu x)^{i\sigma} - (1/2 - i\sigma)(\mu x)^{-i\sigma}] + O(x^{1/2}), \quad x \rightarrow 0.
 \end{aligned}
 \tag{90}$$

The infinite sequence of negative energy levels representing its discrete spectrum is given by

$$E_{\mu,n} = -4\mu^2 \exp \frac{\pi + 2\theta\sigma}{\sigma} \exp \frac{2\pi n}{\sigma}, \quad n \in \mathbb{Z};
 \tag{91}$$

a position of one of the negative energy levels in any of the intervals

$$\left(-4\mu_0^2 e^{\frac{\theta\sigma + \pi}{\sigma}} e^{\frac{2\pi m}{\sigma}}, -4\mu_0^2 e^{\frac{\theta\sigma - \pi}{\sigma}} e^{\frac{2\pi m}{\sigma}}\right), \quad m \in \mathbb{Z},$$

uniquely determines the Hamiltonian $\hat{H}_{4,\mu}$. The complete orthonormalized system (81) and (82) of eigenfunctions for the Hamiltonian $\hat{H}_{4,\mu}$ is written in terms of the scale parameter μ as follows:

$$u_{E_{\mu,n}}(x) = \left(\frac{2|E_{\mu,n}| \sinh(\pi\sigma)}{\pi\sigma}\right)^{1/2} x^{1/2} K_{i\sigma}(|E_{\mu,n}|^{1/2}x),
 \tag{92}$$

$$\begin{aligned}
 u_{4,\mu;E}(x) &= \frac{1}{2\sqrt{\cosh \pi \sigma + \cos \Phi_\mu(E)}} \\
 &\quad \times [e^{i\theta_\sigma} (E/4\mu^2)^{-i\sigma/2} x^{1/2} J_{i\sigma}(\sqrt{E}x) + e^{-i\theta_\sigma} (E/4\mu^2)^{i\sigma/2} x^{1/2} J_{-i\sigma}(\sqrt{E}x)], \\
 \Phi_\mu(E) &= \sigma \ln(E/4\mu^2) - 2\theta_\sigma, \quad E \geq 0.
 \end{aligned}
 \tag{93}$$

The scale parameter μ , as well as μ_0 , is evidently defined modulo the factor $\exp \frac{\pi m}{\sigma}$, $m \in \mathbb{Z}$; the a.b. conditions (90) are invariant under the change $\mu \rightarrow e^{\frac{\pi m}{\sigma}} \mu$; accordingly, the discrete spectrum (91) is also invariant under this change, and the same holds for the normalized eigenfunctions (92) and (93) up to the irrelevant factor -1 in front of eigenfunctions of continuous spectrum for odd m .

All s.a. Calogero Hamiltonians that form a $U(1)$ family for each value of the coupling constant α in all three regions of the values of $\alpha < 3/4$ are thus parameterized by a scale parameter μ , and in the region $-1/4 < \alpha < 3/4$ we must distinguish two different subfamilies by additional indices $+$ or $-$.

We now turn to the problem of the scale symmetry for s.a. Calogero Hamiltonians. The scale symmetry is associated with the one-parameter group of unitary scale transformations $\hat{U}(l)$, $l > 0$, defined by (4). Under a preliminary ‘naive’ treatment of the Calogero problem, see section 2, the ‘naive’ Hamiltonian \hat{H} identified with the initial differential expression (2), which has been considered as a s.a. operator without any reservations about its domain, formally satisfies the scale-symmetry relation (5). It is this relation that is a source of ‘paradoxes’ concerning the spectrum of the ‘naive’ \hat{H} . One of our duties is to resolve these paradoxes.

If we extend relation (5) to the s.a. Calogero Hamiltonians $\hat{H}_{[i]}$, $[i] = 1; 2, \mu, +; 2, \mu, -; 3, \mu; 4, \mu$, we must recognize that this relation is nontrivial because the operators $\hat{H}_{[i]}$ are unbounded, and, in general, their domains $D_{H_{[i]}}$ change with changing the scale parameter μ that naturally changes under scale transformations. The relation

$$\hat{U}^{-1}(l)\hat{H}_{[i]}\hat{U}(l) = l^{-2}\hat{H}_{[i]} \iff \hat{H}_{[i]}\hat{U}(l) = l^{-2}\hat{U}(l)\hat{H}_{[i]}
 \tag{94}$$

for the Hamiltonian $\hat{H}_{[i]}$ with a specific $[i]$, if does hold, implies that, apart from the fact that ‘the rule of action’ of the operator $\hat{H}_{[i]}$ changes in accordance with (94), its domain $D_{H_{[i]}}$ is invariant under scale transformations:

$$\hat{U}(l)D_{H_{[i]}} = D_{H_{[i]}}.
 \tag{95}$$

In such a case, we say that the Hamiltonian $\hat{H}_{[i]}$ is scale covariant and is of scale dimension $d_{H_{[i]}} = -2$; in short, we speak about the scale symmetry of the Hamiltonian $\hat{H}_{[i]}$. If relation (95) does not hold, i.e. if the domain $D_{H_{[i]}}$ of the Hamiltonian $\hat{H}_{[i]}$ is not scale invariant, we are forced to speak about the phenomenon of a spontaneous breaking of scale symmetry for the Hamiltonian $\hat{H}_{[i]}$.

The initial symmetric operator \hat{H} and its adjoint \hat{H}^+ associated with the differential expression (2) and defined on the respective domains D_H (11) and D_{H^+} (12) are scale covariant because both D_H and D_{H^+} are evidently scale invariant. The s.a. extensions $\hat{H}_{[i]}$ of the scale covariant \hat{H} can lose this property. On the other hand, $\hat{H}_{[i]}$ are s.a. restrictions of \hat{H}^+ , and their domains $D_{H_{[i]}}$ belong to the scale-invariant domain D_{H^+} , $D_{H_{[i]}} \subseteq D_{H^+}$. Therefore, the scale symmetry of a specific Hamiltonian $\hat{H}_{[i]}$ is determined by a behavior of the a.b. conditions specifying this s.a. operator and thus restricting its domain in comparison with D_{H^+} under scale transformations. This behavior is different for different $[i]$; namely, it is different for the above four regions of the values of α (see section 3) and strongly depends on the value of the scale parameter μ specifying the s.a. Hamiltonians in each of the last three regions. We consider these four regions sequentially.

(i) First region: $\alpha \geq 3/4$.

For each α in this region, the single s.a. Calogero Hamiltonian \hat{H}_1 coincides with the operator \hat{H}^+ , $\hat{H}_1 = \hat{H}^+$, and is therefore scale covariant,

$$\hat{U}(l)\hat{H}_1\hat{U}^{-1}(l) = l^{-2}\hat{H}_1. \tag{96}$$

In other words, the scale symmetry holds for $\alpha \geq 3/4$. The scale transformation law (4) as applied to eigenfunctions (60) yields

$$u_{1,E}(x) \rightarrow \hat{U}(l)u_{1,E}(x) = l^{-1}u_{1,l^{-2}E}(x), \tag{97}$$

which we treat, in particular, as the scale transformation law for the energy spectrum, given by

$$E \rightarrow l^{-2}E, \tag{98}$$

i.e. the spatial dimension of energy $d_E = -2$. The group of scale transformations acts transitively on the energy spectrum, the semiaxis \mathbb{R}_+ , except the point $E = 0$ that is a stationary point. This coincides with our preliminary expectations in section 2.

(ii) Second region: $-1/4 < \alpha < 3/4$.

The change of a.b. conditions (83) under scale transformations (4) is given by the natural scale transformation

$$\mu \rightarrow l^{-1}\mu, \tag{99}$$

of the dimensional scale parameter μ (its spatial dimension is -1), or, in terms of the dimensionless extension parameter λ , by

$$\lambda \rightarrow l^{2\sigma}\lambda, \tag{100}$$

which implies that under the scale transformations the respective domain $D_{H_{2,\mu,+}}$ of the Hamiltonian $\hat{H}_{2,\mu,+}$, $0 < \mu < \infty$, transforms to $D_{H_{2,l^{-1}\mu,+}}$,

$$D_{H_{2,\mu,+}} \rightarrow \hat{U}(l)D_{H_{2,\mu,+}} = D_{H_{2,l^{-1}\mu,+}}. \tag{101}$$

It follows that the scale transformations change the Hamiltonian $\hat{H}_{2,\mu,+}$ to another Hamiltonian $\hat{H}_{2,l^{-1}\mu,+}$,

$$\hat{H}_{2,\mu,+} \rightarrow \hat{U}(l)\hat{H}_{2,\mu,+}\hat{U}^{-1}(l) = l^{-2}\hat{H}_{2,l^{-1}\mu,+}, \tag{102}$$

which means that the scale symmetry is spontaneously broken for the Hamiltonians $\hat{H}_{2,\mu,+}$, $0 < \mu < \infty$. The scale transformation law for the eigenfunctions (84) is given by

$$u_{2,\mu,+;E}(x) \rightarrow \hat{U}(l)u_{2,\mu,+;E}(x) = l^{-1}u_{2,l^{-1}\mu,+;l^{-2}E}(x), \quad E \geq 0. \tag{103}$$

The same evidently holds true for the Hamiltonians $\hat{H}_{2,\mu,-}$, $0 < \mu < \infty$, specified by a.b. conditions (85): the respective formulas (99) and (100) remain unchanged, while in formulas (101)–(103) the subscript $+$ changes to the subscript $-$, and formula (103) for the eigenfunctions of the continuous spectrum is supplemented by the formula for bound-state eigenfunction (86) and (87)

$$u_{E_{2,\mu,-}}(x) \rightarrow \hat{U}(l)u_{E_{2,\mu,-}}(x) = u_{E_{2,l^{-1}\mu,-}}(x), \quad E_{2,l^{-1}\mu,-} = l^{-2}E_{2,\mu,-}. \tag{104}$$

The Hamiltonians $\hat{H}_{2,\infty}$ and $\hat{H}_{2,0}$ corresponding to the respective exceptional values $\mu = \infty$ ($\lambda = 0$) and $\mu = 0$ ($\lambda = \infty$) of the scale parameter μ and specified by the respective a.b. conditions

$$\psi_{2,\infty}(x) = cx^{1/2+x} + O(x^{3/2}), \quad x \rightarrow 0, \quad \psi_{2,0}(x) = cx^{1/2-x} + O(x^{3/2}), \quad x \rightarrow 0,$$

are scale covariant, which means that copies of formulas (96)–(98) with replacing subscript 1 to the respective subscripts 2, ∞ and 2, 0 hold true. If we require scale symmetry in the Calogero problem, then only the two possibilities, $\hat{H}_{2,\infty}$ or $\hat{H}_{2,0}$, remain for the s.a. Calogero Hamiltonian in the interval $-1/4 < \alpha < 3/4$. We note that this interval of α includes the point $\alpha = 0$ ($\kappa = 1/2$) corresponding to a free motion. Therefore, all the above-said concerning the spontaneous scale-symmetry breaking relates to the case of a free particle on a semiaxis.

(iii) Third region: $\alpha = -1/4$.

The change of the a.b. conditions (88) under the scale transformations (4) is equivalent to rescaling (99) of the dimensional parameter μ , or to the change $\lambda \rightarrow \lambda - \ln l$ of the dimensionless extension parameter λ . A further consideration is completely similar to the preceding one to yield that copies of relations (101)–(104), with the subscript 2 replaced by the subscript 3, and with the subscripts + and – eliminated, hold true for the Hamiltonians $\hat{H}_{3,\mu}$, $0 < \mu < \infty$, which implies scale-symmetry breaking for these Hamiltonians.

As to the Hamiltonian \hat{H}_3 corresponding to the exceptional values $\mu = 0$ and $\mu = \infty$ of the scale parameter μ , which are equivalent, $0 \sim \infty$, and specified by the a.b. conditions $\psi_3(x) = cx^{1/2} + O(x^{3/2})$, this Hamiltonian is scale covariant, and copies of relations (96), (97), and (98) with the substitution $1 \rightarrow 3$ hold true. If we require scale symmetry for the s.a. Calogero Hamiltonian with $\alpha = -1/4$, then it is only the Hamiltonian \hat{H}_3 that survives.

(iv) Fourth region: $\alpha < -1/4$.

The change of the a.b. conditions (90) under the scale transformations (4) is equivalent to a modified rescaling $\mu \rightarrow l^{-1}\mu \exp \pi m/\sigma$, of the dimensional extension parameter μ , where an integer m is defined by the condition $\mu_0 \leq l^{-1}\mu \exp \pi m/\sigma < \mu_0 \exp \pi/\sigma$: the changed μ must remain within the interval $[\mu_0, \mu_0 \exp \pi/\sigma)$, see (89); this is equivalent to the change $\theta \rightarrow (\theta + \sigma \ln l) |_{\text{mod } \pi}$ of the dimensionless extension parameter θ . It follows that for the Hamiltonians $\hat{H}_{4,\mu}$, $\mu_0 \leq \mu \leq \mu_0 e^{\pi/\sigma}$, $\mu_0 \sim \mu_0 e^{\pi/\sigma}$, the relations

$$\begin{aligned} D_{H_{4,\mu}} &\rightarrow \hat{U}(l) D_{H_{4,\mu}} = D_{H_{4,\mu l^{-1} \exp \pi m/\sigma}}, \\ \hat{H}_{4,\mu} &\rightarrow \hat{U}(l) \hat{H}_{4,\mu} \hat{U}^{-1}(l) = l^{-2} \hat{H}_{4,\mu l^{-1} \exp \pi m/\sigma}, \\ u_{E_{\mu,n}}(x) &\rightarrow \hat{U}(l) u_{E_{\mu,n}}(x) = u_{E_{\mu l^{-1} \exp \pi m/\sigma, n-m}}(x), E_{\mu l^{-1} \exp \pi m/\sigma, n-m} = l^{-2} E_{\mu,n}, \\ u_{4,\mu;E}(x) &\rightarrow \hat{U}(l) u_{4,\mu;E}(x) = (-1)^m l^{-1} u_{4,\mu l^{-1} \exp \pi m/\sigma; l^{-2} E}(x) \end{aligned}$$

hold true.

This means that the scale symmetry is spontaneously broken for $\hat{H}_{4,\mu}$. The peculiar feature of the fourth region is that for $l = \exp \pi n/\sigma$, $n \in \mathbb{Z}$, the scale symmetry holds true. In other words, the scale symmetry is not broken completely, but up to an infinite cyclic subgroup. In particular, this subgroup acts transitively on the discrete energy spectrum.

This is the fate of the scale symmetry in the QM Calogero problem.

The paradoxes concerning the scale symmetry in the Calogero problem and considered in section 2 are thus resolved. Namely, in general, there is no scale symmetry in the problem for $\alpha < 3/4$. In the latter case, the ‘naive’ Calogero Hamiltonian \hat{H} of section 2 is actually the operator \hat{H}^+ that is scale covariant but not s.a.. As for s.a. Calogero Hamiltonians, all possibilities for a negative part of the energy spectrum considered in section 2 are generally realized by different Hamiltonians specified by different a.b. conditions. In general, the scale symmetry shifts energy levels together with Hamiltonians.

We conclude the above consideration with the following remarks for physicists.

We have a unique QM description of a nonrelativistic particle moving on a semiaxis in the Calogero potential (1) with the coupling constant $\alpha \geq 3/4$. In the case of $\alpha < 3/4$, mathematics presents different possibilities related to different admissible s.a. asymptotic boundary conditions at the origin that are specified in terms of the scale parameter μ . But a final choice, which is reduced to a specific choice of the scale parameter μ , belongs to the physicist.

The origin of this parameter presents a physical problem, as well as the physical interpretation of the chosen s.a. Hamiltonian, as a whole. We can only note that the usual regularization (3) of the Calogero potential by a cut-off at a finite radius and the consequent passage to the limit of zero radius yields $\mu = \infty$ in the case of $-1/4 \leq \alpha < 3/4$; a peculiar feature of the case of $\alpha = -1/4$ is that $\mu = \infty$ is equivalent to $\mu = 0$. Such a choice of the scale parameter corresponds to the minimum possible singularity of wavefunctions in the s.a. Hamiltonian domain, including eigenfunctions, at the origin. In the case of $\alpha < -1/4$, the regularization procedure does not provide any answer: the zero-radius limit does not exist. A suggestion on the nature of the scale parameter μ , $0 \leq \mu < \infty$, in the case of $-1/4 < \alpha < 3/4$, $0 < \mu < \infty$ in the case of $\alpha = -1/4$, and $\mu_0 \leq \mu \leq \mu_0 \exp \pi/\sigma$ in the case of $\alpha < -1/4$, has been presented above in section 3: it is conceivable that this parameter is a manifestation of an additional δ -like term in the potential.

When deciding on a specific value of the scale parameter μ , one of the additional arguments can be related to scale symmetry. In the case of $\alpha \geq 3/4$, scale symmetry holds true. In the case of $-1/4 \leq \alpha < 3/4$, scale symmetry is spontaneously broken for a generic μ . As for any spontaneously broken symmetry, scale symmetry does not disappear but transforms one physical system to another nonequivalent physical system. But if we require scale symmetry, as we do in similar situations with rotation symmetry or reflection symmetry, then a possible choice strongly narrows to $\mu = \infty$ (the minimum possible singularity of wavefunctions at the origin) or $\mu = 0$ (the maximum possible singularity) in the case of $-1/4 < \alpha < 3/4$ and to $\mu = \infty \sim \mu = 0$ (the minimum possible singularity) in the case of $\alpha = -1/4$. For strongly attractive Calogero potentials with $\alpha < -1/4$, the requirement of scale symmetry cannot be fulfilled: scale symmetry is spontaneously broken for any μ .

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